INTRODUCTION

Several field theory models on κ space time have been constructed using different techniques such as by F. Mejer and J Lukierski specially in, scalar field theory on κ-Minkowski space. It has been shown by S. Tanimutra that in the flat space time, Feynman approach and minimal coupling method are equivalent which is useful to derive the general equation of motion for a charged particle. We discuss here the applicability of Feynman approach for the case of general relativity, resulting in the derivation of the geodesic equation. We generalize the procedure for κ-Minkowski space. We obtain here corrections to the geodesic equation due to the κ-deformation of space time, up to the first order in the deformation parameter. We know that a relativistic particle of mass m and electric charge e is described by in 4D-Minkowski space, where $x_\mu(\tau)$ in 4D- Minkowski space, where $\tau$ is a parameter. Let us write the following relations

\[
\left[ x_\mu (\tau), x_\nu (\tau) \right] = 0, \left[ x_\mu (\tau), p_\nu (\tau) \right] = -i \eta_{\mu \nu},
\]
\[
F_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu - i \epsilon_{\mu \nu \rho \sigma} F^\rho F^\sigma,
\]

Where $p_\mu = mx_\mu^\prime + A_\mu$ is the canonical momentum operator, $F_\mu = m x_\nu$ the force, $F_\mu$ the electromagnetic strength tensor, $A_\mu$ is a gauge field and $\varphi(x)$ is an arbitrary function of $x$. We deal with gravity only, which do not require the gauge field. We only evaluate particles with no electric charge, which for a neutral particle, we obtain the following equations.

\[
\left[ x_\mu , x_\nu \right] = 0, \left[ p_\mu , p_\nu \right] = 0, \left[ x_\mu , p_\nu \right] = -i \eta_{\mu \nu},
\]
\[
F_\mu = 0, p_\mu = m x_\mu^\prime.
\]

$\varphi(x) = 0$, to find the correct geodesic equation. M. Montesinos obtained that the generalization from flat to curved space by taking
Eq. (1) as valid in a local Lorenz frame of reference and effect of gravity is brought in by replacing the Minkowskian metric $\eta_{\mu\nu}$ with an arbitrary metric $g_{\mu\nu}(X)$. He has shown that this assumption leads to geodesic equation. We choose

$$[X_{\mu}, X_{\nu}] = 0 \quad [X_{\mu}, P_{\nu}] = -i g_{\mu\nu}(X),$$

Let us assume that the 'metric' $g_{\mu\nu}(X)$ is a function of operator $X$ which is a symmetric tensor.

**Theorem (Generalization of commutative space time)**

Let $X_{\alpha}(t)$ be a new position operator and $p_{\alpha}(t)$ is the corresponding conjugate momenta, $mX_{\mu} = P_{\mu}$, then solution of equation may be obtain in terms of the operators given in (2).

**Proof**

We construct operators X and P expressed as follows.

$$X_{\mu} = x_{\mu}, \quad P_{\mu} = \partial_{\mu} p^{\alpha} \quad \text{...(4)}$$

Where $x_{\mu}$ and $p_{\nu}$ satisfy relation. By taking the derivative with respect to of Eqs.(4), we obtain the following relation.

$$X_{\mu} = x_{\mu}, \quad P_{\mu} = \partial_{\mu} p^{\alpha} \quad \text{...(5)}$$

Where $P_{\mu} = mX_{\mu}$. Combining equations (5) and (2) we obtain

$$\left[ P_{\mu}, P_{\nu} \right] = i \left( \partial_{\mu} \frac{\partial \phi_{\beta}}{\partial x_{\alpha}} - \partial_{\nu} \frac{\partial \phi_{\beta}}{\partial x_{\alpha}} \right) p^{\beta}, \quad \text{...(6)}$$

Where $\left[ p_{\mu}, x_{\nu} \right] = i \frac{\partial \phi_{\mu}}{\partial x_{\nu}}$. Also we get

$$\partial_{\mu} x_{\nu} = \frac{\partial \phi_{\mu}}{\partial x_{\nu}} X_{\beta} = \frac{i}{m} \frac{\partial \phi_{\mu}}{\partial x_{\alpha}} P_{\alpha} = \frac{1}{m} \frac{\partial \phi_{\mu}}{\partial x_{\alpha}} g_{\beta\alpha} p^{\alpha}$$

Combining equations (3), (5), (6), we obtain the equation

$$\left[ X_{\nu}, mX_{\mu} \right] = -i \left( \partial_{\mu} \frac{\partial \phi_{\beta}}{\partial x_{\alpha}} - \partial_{\nu} \frac{\partial \phi_{\beta}}{\partial x_{\alpha}} \right) p^{\beta}, \quad \text{...(7)}$$

By choosing the operator $X$ such that is

$$X_{\mu} = X_{\mu}(x, p), \text{LHS of (7) reduces to the equation}$$

$$\left[ X_{\nu} mX_{\mu} \right] = \left[ x_{\nu}, \dot{x}_{\mu} \right] = -i \frac{\partial \dot{x}_{\nu}}{\partial p^{\mu}}$$

We now integrate Eq. (6) over $p^{\mu}$ which gives the relation

$$mX_{\nu} = \frac{1}{2} \left( \partial_{\mu} \frac{\partial \phi_{\beta}}{\partial x_{\alpha}} - \partial_{\nu} \frac{\partial \phi_{\beta}}{\partial x_{\alpha}} \right) p^{\beta}$$

By taking $G_{\nu}(x) = 0$ we get

$$\frac{1}{m} \frac{\partial \phi_{\mu}}{\partial x_{\nu}} P_{\alpha} = 0 \quad \text{...(8)}$$

Where all Jacobi identities are satisfied. Eq. (6) is similar to the Christoffel symbol in general relativity, and Eq. (8) is similar to the geodesic equation. Let us assume that 'metric' is invertible and define an inverse of the symmetric tensor by the following relation

$$\partial_{\mu} \partial_{\nu} x^{\alpha} = \partial_{\nu} x^{\alpha}, \quad \text{...(9)}$$

Combining the equations (4), (7) and (9) we get

$$\partial_{\mu} \partial_{\nu} x^{\alpha} = \partial_{\nu} x^{\alpha}$$

Combining the geodesic equation and Christoffel symbol, we get the geodesic equation

$$\ddot{x}_{\nu} + \Gamma_{\nu}^{\mu\beta} \dot{x}_{\mu} \dot{x}_{\beta} = 0, \quad \text{...(10)}$$
Where tensor is treated only as a symmetric tensor with an inverse defined in Eq. (10).

**Theorem (Derivation of geodric equation in κ-Minkowski space time)**

We derive the geodesic equation for a particle moving in the non-commutative curved space time and analyze the κ-Minkowski deformations of gravity. The methods taken here use κ-Minkowskideformations on the flat space time. It is further generalized to κ-deformed space time with arbitrary metric.

**Proof**

Let us define κ-Minkowski space by the relation

\[ \hat{x}_\mu = x_\mu \phi^\alpha_\mu (p), \]

where \( \phi^\alpha_\mu (p) \) satisfy the identity

\[ \frac{\partial \phi^\alpha_\mu}{\partial p^\alpha} \phi^\beta_\nu - \frac{\partial \phi^\beta_\nu}{\partial p^\alpha} \phi^\alpha_\mu = a_\mu \phi^\alpha_\nu - a_\nu \phi^\alpha_\mu \ldots (12) \]

Let us solve Eq. (11) up to the first order in deformation parameter \( a \), which gives

\[ \phi^\alpha_\mu = \delta^\alpha_\mu \left[ 1 + \alpha (a, p) \right] + \beta p^\alpha_\mu + \gamma \phi^\alpha_\mu a_\mu, \]

\[ \alpha, \beta, \gamma \in \mathbb{R} \]

where parameters of the realization \( \alpha, \beta, \gamma \) satisfy the a constraint

\[ \gamma - \alpha = 1, \]

Let us define an operator \( \hat{y} \) which commutes with \( \hat{x}_\mu \), i.e.

\[ \left[ \hat{y}_\mu, \hat{x}_\nu \right] = 0 \iff \left[ \hat{y}_\mu, \hat{x}_\nu \right] = -i(a_\mu \hat{y}_\nu - a_\nu \hat{y}_\mu) \]

Any function of also commutes with

\[ \left[ f(\hat{y}), \hat{x}_\mu \right] = 0 \ldots (13) \]

Where we take only first order in the modified form of give \( \hat{y} \) and \( f(\hat{y}) \) up to the first order in

\[ \hat{y}_\mu = x_\mu + \gamma x_\nu (a, p) + (\gamma - 1) x_\nu (a, p) a_\mu + \beta (x, a) p_\mu \]

\[ f(\hat{y}) = f(x) + \gamma (\frac{df}{dx} x_\nu) a_\mu + (\gamma - 1) (x_\nu \frac{df}{dx} x_\nu) p_\mu + \beta (x, a) x_\nu \frac{df}{dx} \]

The canonical momentum operator (in e = 0 case ) \( \hat{p}_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} \) as derived by the relation

\[ \hat{p}_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} \hat{p}_\mu = p_\mu + (a + \beta \alpha (a, p)) p_\mu + \gamma a_\mu p_\mu \]

\[ \mathcal{L} \in \mathbb{R} \]

\[ \left[ \hat{p}_\mu, \hat{x}_\nu \right] = i \eta_{\mu\nu} (1 + s (a, p)) + i (s + 2) a_\mu a_\nu + i (s + 1) a_\nu p_\mu \]

\[ s = 2 \alpha + \beta \ldots (14) \]

Hence, we conclude that construction due to E. HariKumar via Feynman approach satisfy all Jacobi identities. The condition that comes by taking the derivative of Eq. (12) with respect to \( \tau \) is

\[ \left[ \hat{p}_\mu, \hat{x}_\nu \right] + \left[ \hat{y}_\mu, \hat{p}_\nu \right] = i (a_\mu \hat{p}_\nu - a_\nu \hat{p}_\mu). \]

Hence the geodesic equation in the κ-Minkowski space time is established.

**Theorem (κ-dependent corrections to the geodesic equation)**

**Proof**

Let us choose the conjugate pairs \((x, p)\) and for flat non-commutative space time, \((\hat{x}, \hat{p})\) in commutative and non-commutative space respectively. It is shown that all the operators in the flat non-commutative space time are expressed in
terms of $x$, $p$ and deformation parameter $a$. For the non-commutative space time with curvature. Let us construct it as functions of $x$, $p$ and deformation parameter. In case of neutral particles, conjugate momenta is given by $\hat{P}_\mu = \frac{d\hat{x}_\mu}{d\tau}$. We derive the corrections to the geodesic equation due to the $\kappa$-deformation of space time. Let us consider the relation.

$$[X_\mu, X_\nu] = i(a_\mu X_\nu - a_\nu X_\mu)$$

Where

$$\hat{X}_\mu = X_\alpha \phi^\alpha_\mu$$

And $\phi^\alpha_\mu$ satisfies Eq. (12). $P_\mu$ satisfy all the Jacobi Indentifies and the relation

$$[P_\mu, X_\nu] + [\hat{X}_\mu, P_\nu] = i(a_\mu P_\nu - a_\nu P_\mu).$$

But in the limit $a \to 0$, it implies that

$$\hat{X}_\mu \to X_\mu = x_\mu \quad \quad \hat{P}_\mu \to P_\mu = \phi_\mu(\hat{x})$$

$$[\hat{X}_\mu, P_\nu] \to \phi_{\mu\nu}(x)$$

By taking limit $\phi_{\mu\nu}(x) \to \eta_{\mu\nu}$ we get the following relations

$$\hat{X}_\mu \to \hat{x}_\mu = x_\mu \phi^\alpha_\mu \quad \quad \hat{P}_\mu \to \hat{P}_\mu = P_\mu \phi^\alpha_\mu$$

$$[\hat{X}_\mu, P_\nu] \to [\hat{x}_\mu, \hat{P}_\nu]$$

Combining these equation, and comparing it with (13) & (14) $\hat{P}_\mu$ substitute $\phi_{\alpha\beta}(\hat{x})$ with a function that commutes with $\hat{X}_\mu$. That is with $\phi_{\alpha\beta}(\hat{y})$. Thus, we obtain the required form

$$\hat{P}_\mu = \phi_{\alpha\beta}(\hat{y}) p^\beta \phi^\alpha_\mu$$

By an appropriate application of (14), we find that this construction satisfies all Jacobi identities and Eq. (15) is also satisfied to all orders in $a$. $\hat{P}_\mu$. Hence, get its modified form as follows.

$$\hat{X} = x_\alpha \phi^\alpha_\mu \quad \quad \hat{P}_\mu = \phi_{\alpha\beta}(\hat{y}) p^\beta \phi^\alpha_\mu$$

$$[\hat{X}_\mu, X_\nu] = i(a_\mu X_\nu - a_\nu X_\mu)$$

An alternative construction of the operator up to the first order in the deformation parameter may be obtained by differentiating Eq. (17) with respect to $\tau$.

$$[\hat{P}_\mu, \hat{X}_\nu] + [\hat{X}_\mu, \hat{P}_\nu] = i(a_\mu \hat{P}_\nu - a_\nu \hat{P}_\mu)$$

It satisfies antisymmetric part of $[\hat{P}_\mu, \hat{X}_\nu]$. We write it as follows

$$\hat{P}_\mu$$

$$[\hat{P}_\mu, \hat{X}_\nu] = \hat{S}_{\mu\nu} A_{\mu\nu}$$

Where $\hat{S}_{\mu\nu} = \hat{S}_{\nu\mu}$ and $\hat{A}_{\mu\nu} = \hat{A}_{\nu\mu}$. Combining Eq. (16) and (17), We get

$$\hat{A}_{\mu\nu} = \frac{i}{2} (a_\mu \hat{P}_\nu - a_\nu \hat{P}_\mu)$$

By taking the limit $a \to 0$, we obtain

$$[\hat{P}_\mu, \hat{X}_\nu] \to \hat{A}_{\mu\nu} = \hat{S}_{\mu\nu} A_{\mu\nu}$$

Hence, the case of up to the first order in the deformation parameter $a_i.e.$

$$\hat{S}_{\mu\nu} = i \phi_{\mu\nu} + i \phi_{\alpha\beta} G_{\mu\nu}(\hat{x}) p_\beta + O(a^2)$$

Here $G_{\alpha\beta} = G_{\alpha\beta}$ and we get the following relations.
\[ [\hat{P}_\mu, \hat{X}_\nu] = i\hat{\theta}_\mu + i\alpha \hat{G}^{\mu\nu}(x)p_\mu + \frac{i}{2}(\alpha \hat{P}_\mu - \alpha \hat{P}_\mu) \]

We get constraints on \( \hat{G}^{\mu\nu}(x) \) that the Jacobi identities must be satisfied up to the first order in \( a \).

\[ \Rightarrow [[\hat{X}_\mu, \hat{X}_\nu], \hat{P}_\lambda] + [[\hat{X}_\mu, \hat{P}_\lambda], \hat{X}_\nu] + [[\hat{P}_\lambda, \hat{X}_\mu], \hat{X}_\nu] = 0 \]

We get
\[
\alpha (\hat{G}^{\mu\nu}_{\alpha \beta} - \hat{G}^{\mu\nu}_{\beta \alpha}) = \\
\alpha a \left( x^\lambda \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\lambda} \right) + \beta a \left( a - x^\lambda \frac{\partial}{\partial x^\nu} - a - x^\nu \frac{\partial}{\partial x^\lambda} \right) \\
+ \gamma(x, \alpha) \left( \frac{\partial x^\lambda}{\partial x^\nu} - \frac{\partial x^\nu}{\partial x^\lambda} \right) + \frac{3}{2} (\alpha - x^\lambda \frac{\partial}{\partial x^\nu} - a - x^\nu \frac{\partial}{\partial x^\lambda})
\]

Let us construct \( \hat{G}^{\mu\nu}(x) \), from \( \eta_{\mu\nu}, e_{\mu\nu} \) and \( \frac{\partial x^\mu}{\partial \xi} \eta_{\mu\nu} \), and express it as

\[ \hat{G}^{\mu\nu}_{\alpha\beta} = \sum \mathcal{A}_i (\theta, \eta)_{\alpha\beta} + \sum \mathcal{B}_i (\eta, \theta)_{\alpha\beta} \]

Hence \( \hat{G}^{\mu\nu}_{\alpha\beta} \) is determined by the parameters \( \alpha \) and \( \beta \) and four more free parameters. This general used form but valid up to the first order in the deformation parameter \( a \). The construction \( \hat{p}_\mu = \partial_g(x) p^{\mu\nu} \) where is the special case of this general procedure.

**CONCLUSION**

1. The principal characteristic of this approach is that all the corrections depend on the choice of realization of the parameters on the mass of the test particle.

2. We analysed here the \( a \)-dependent correction to the Newtonian limit of the geodesic equation which shows that the Newtonian force/potential remains radial, but depends on the mass of the test particle.

3. We have shown here that the \( \kappa \)-deformed commutation relations between phase space variables induce modified uncertainty relations.

4. It has been shown that the commutative limit \( [\hat{X}_{\mu\nu}] \) gives rise to the metric \( e_{\mu\nu} \) and by analogy, we interpret \( [\hat{X}_{\mu\nu}] \) non-commutative metric \( e_{\mu\nu} \).

**REFERENCES**