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Self-duality and generalized Bicrossproducts Hopf algebras

M. M. AL-MOSA AL-SHOMRANI

Department of Mathematics, Faculty of Science, King Abdulaziz University,
Jeddah, P. O. Box 80257, Jeddah 215 89 (Saudi Arabia).

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ABSTRACT

In this paper we generalize the construction of a bicrossproduct Hopf algebra from a factorization of a finite group X into a subgroup G and a subsemigroup H. In addition, we show that these bicrossproduct Hopf algebras are self-dual as Hopf algebras whenever they correspond to factor-reversing automorphisms of X.

Key words: Self duality, bicrossproducts Hopf algebras.

INTRODUCTION

Group factorizations are very common in mathematics. Among their uses is the bicrossproduct construction which is one of the primary sources of non-commutative and non-cocommutative Hopf algebras. These bicrossproduct Hopf algebras have been introduced by Majid[10] and Takeuchi[16]. Since then, bicrossproduct Hopf algebras have been extensively studied[2-4,6,9]. These algebras have many applications, for example Majid in[10] showed that they can be considered as a systems combine quantum mechanics with geometry[11].

In 1996, Beggs *et al*.,[5] have computed the quantum double construction of Drinfeld[7] for the bicrossproduct Hopf algebra associated to the factorization X = GM, where G and M are subgroups of the group X, which led to an interesting generalization of crossed modules to bicrossed bimodules. In addition, they showed that basis-preserving selfduality structures for the bicrossproduct Hopf algebras are in one-to-one correspondence with factor-reversing group isomorphisms.

In this paper we show that Hopf algebras can be constructed by using more general factorizations of finite groups. More specifically, we show that the bicrossproduct Hopf algebras can be associated to a factorization X = GH, where G is a subgroup of the group X and H is a subsemigroup of X. In addition, it is shown that basis-preserving self-duality structures for these bicrossproduct Hopf algebras are in one-to-one correspondence with factor-reversing semigroup isomorphisms.

Throughout this paper we assume that all groups mentioned, unless otherwise stated, are finite and that all vector spaces are finite dimensional over a general field k . The conventions and notation are mainly taken from[5]. The reader is referred to[12-15] for the basic results of Hopf algebras.

Preliminaries

Let *k* be a field and G a semigroup with identity. Denote the k-vector space generated by G by kG. Defining the multiplication on kG by

$$\left(\sum_{x\in G} a_x x\right)\left(\sum_{y\in G} b_y y\right) = \sum_{z\in G}\left(\sum_{xy=z} a_x b_y\right) z,$$

kG becomes a ring. The map ηkG : k→ kG given by ηkG(a) = a1 where 1 is the identity element of G makes kG a k- algebra. The k-algebra kG is said to be the semigroup k-algebra of G[1].

Let X = GM be a group which factorizes into two subgroups G and M. Then each group acts on the other through left and right actions: ▷ M× G → G and : M×G →M defined by su = (s u)(s u), where u ∈ G and s ∈ M. These actions obeying the following conditions for all s, t ∈ M and u, v ∈ G[5]:

s e = s, (s u) v = s (uv);
e u = e, (st) u = (s (t u))(t u)

e u = u, s (t u) = (st) u;
s e = e, s (uv) = (s u)((s u) v)) ...(1)

Associating to this factorization, we can define the bicrossproduct Hopf algebra H =kM k(G) with basis $s\otimes\delta_u$ where s ∈ M and u ∈ G. The product, unit, coproduct, counit and antipode are defined as follows[5]:

$$(s\otimes\delta_u)(t\otimes\delta_v)=\delta_{u,t\triangleright v}(st\otimes\delta_v),\ 1_H=\sum_u e\otimes\delta_u,$$

$$\Delta(s\otimes\delta_u)=\sum_{x,y,\in G;xy=u}(s\otimes\delta_x\otimes(s\triangleleft x)\otimes\delta_y;\ \epsilon_H=(s\otimes\delta_u)=\delta_{u,e},$$

$$S(s\otimes\delta_u)=(s\triangleleft\upsilon)^{-1}\otimes\delta_{(s\triangleright\upsilon)^{-1}}$$

Also, we can define the dual of $\mathcal{H}$ which is again a bicrossproduct Hopf algebra $\mathcal{H}$ *=k(M) kG with basis $\delta_s\otimes u$ where s ∈ M and u ∈ G. The product, unit, coproduct, counit and antipode are defined as follows[5]

$$(\delta_s\otimes u)(\delta_t\otimes\upsilon)=\delta_{s\triangleleft u,t}(\delta_s\otimes u\upsilon),\ 1_{H^*}=\sum_s\delta_s\otimes e,$$

$$\Delta(\delta_s\otimes u)=\sum_{m,n\in M;mn=s}\delta_m\otimes(n\triangleright u)\otimes\delta_n\otimes_u;\ \epsilon_{H^*}(\delta_s\otimes u)=\delta_{s,e},$$

$$S(\delta_s\otimes u)=\delta_{(s\triangleleft u)^{-1}}\otimes_{(s\triangleright u)^{-1}}$$

Self-duality of bicrossproducts

Here we study the bicrossproduct Hopf algebras associated to a factorization of a group into a subgroup and a semisubgroup with identity and a left inverse property. This may have some relevance to the work of Green, Nichols and Taft[8] concerning one sided Hopf algebras structures. If it exists, the left inverse for an element a ∈ H will be denoted by a^L.

Let X = GH be a group which factorizes into a subgroups G and a semisubgroupwith identity H. Then H acts on G through the right action : H × G→G and G acts on H through the left action :H × G→ H. These actions are defined by au = (a u)(a u), where g ∈ G and a ∈ H. It is easy to show that these actions obeying the following conditions for all a, b ∈ H and μ , v ∈ G

a e = a, (a u) υ = a (uv);
e u = e, (ab) u = (a (b u))(b u)
e u = u, a (b u) = (ab) u;
a e = e, a (uv) = (a u)((a u) υ) ...(2)

It can be seen that we can associate to this factorization a bicrossproduct bialgebra H = kH k(G) with basis a⊗δu where a ∈ H and u ∈ G. ⋈◁The product, unit, coproduct and counit are defined as follows:

$$(a\otimes\delta_u)(b\otimes\delta_u)=\delta_{u,b\triangleright v}(ab\otimes\delta_v),\ 1_H=\sum_u e\otimes\delta_u,$$

$$\Delta(a\otimes\delta_u)=\sum_{x,y,\in G;xy=u}a\otimes\delta_x\otimes(a\triangleleft x)\otimes\delta_y;\ \epsilon_H=(a\otimes\delta_u)=\delta_{u,e},$$

If H posses a left inverse a^L for each a ∈ H, then $\mathcal{H}$ becomes a Hopf algebra and the antipode will be given by:

$$S(a\otimes\delta_u)\ =\ (a\triangleleft u)^L\otimes\delta_{(a\triangleright u)^{-1}}$$

Due to these formulas, it can be noted that $\mathcal{H}$ = kH k(G) has the smash product algebra structure by the induced action of H and the smash coproduct coalgebra structure by the induced coaction of G.

In the symbol $\mathcal{H}$ = kH k(G), kH is the semigroup Hopf algebra of the semigroup H with identity and left inverse property. A basis of kH is given by the elements of H, with multiplication given by the semigroup product in H , and comultiplication given by $\Delta a = a\otimes a$ for $a \in H$. Also, k(G) is the Hopf algebra of functions on G with basis given by δ_u for $u \in G$. The product is just multiplication of functions, and the coproduct is

$$\Delta\delta_u = \sum_{x,y\in G:xy=u} \delta_x \otimes \delta_y$$

Moreover, the part means that kH acts on k(G), and the part means that k(G) coacts on kH.

In addition, a dual bicrossproduct bialgebra $\mathcal{H}^* = k(H)$ kG can be defined with basis $\delta_a\otimes u$ where $a\in H$ and $u\in G$. The product, unit, coproduct and counit are defined as follows:

$$(\delta_a\otimes u)(\delta_b\otimes \upsilon)=\delta_{a\triangleleft u,b}(\delta_a\otimes u\upsilon),\ 1_{H^*}=\sum_a \delta_a\otimes e,$$

$$\Delta(\delta_a\otimes u)=\sum_{m,n\in M;mn=a}\delta_m\otimes(n\triangleright u)\otimes\delta_n\otimes u,\ \ \epsilon_{H^*}(\delta_a\otimes u)=\delta_{a,e},$$

If H posses the left inverse property for each $a\in H$, then $\mathcal{H}^*$ becomes a Hopf algebra and the antipode will be given by:

$$S(\delta_a\otimes u) \;=\; \delta_{(a\triangleleft u)^L}\otimes(a\triangleright u)^{-1}$$

It can be noted that what has been said about H, can be dually said about $\mathcal{H}^*$.

Definition 3.1

Let X be a finite group and X = GH be factorization of X into two subsemigroups G and H with identities. A semigroup isomorphism $f : X\to X$ is defined to be factor-reversing if $f(g)\in H$ for all $g\in G$ and $f(a)\in G$ for all $a\in H$. We need the following lemmas:

Lemma 3.2

Let X = GH be factorization of a group X into a subgroup G and a subsemigroup H with identity. Then for the algebra $\mathcal{H} = kH$ k(G) ,where k(G) is the algebra of function on G and kH is the semigroup algebra of H , an algebra homomorphism: $\mathcal{H}\to\mathcal{H}^*$ which sends basis elements to basis elements can be constructed from afactor-reversing isomorphism of X = GH.

Proof

We Suppose that f is a semigroup isomorphism and we define a linear map f : $\mathcal{H}\to\mathcal{H}^*$ by

$$\tilde{f}(a\otimes\delta_u)=\delta_{f(a\triangleright u)}\otimes f(a\triangleleft u) \qquad ...(3)$$

We want to prove that is an algebra homomorphism. As f is a semigroup homomorphism, we should have $f(b\upsilon)=f(b)f(\upsilon)$ and also $f(b\upsilon) = f((b\ \ \upsilon)\ (b\ \ \upsilon)) = f(b\ \ \upsilon)f(b\ \ \upsilon)$, for all $b\in H$ and $\upsilon\in G$. Thus

$f(b)f(\upsilon) = f(b\ \ \upsilon)f(b\ \ \upsilon)$

$=(f(b\ \ \upsilon)\ \ f(b\ \ \upsilon))(f(b\ \ \upsilon)\ \ f(b\ \ \upsilon))$.

By the uniqueness of factorization, we have

$f(b) = f(b\ \ \upsilon)\ \ f(b\ \ \upsilon)$ ...(4)

$f(\upsilon) = f(b\ \ \upsilon)$ C $f(b\ \ \upsilon)$ ...(5)

Now to prove that is an algebra homomorphism, we show that $((a\otimes\delta_u)(b\otimes\delta_\upsilon)) = (a\otimes\delta_u)\ \ (b\otimes\delta_\upsilon)$ for $a\otimes\delta_u\ b\otimes\delta_u\in H$, $a, b \in\mathcal{H}$ and u, $\upsilon \in G$. We start with the left hand side as follows:

$$\tilde{f}((a\otimes\delta_u)(b\otimes\delta_\upsilon))=\tilde{f}(\delta_{u,b\triangleright\upsilon}(ab\otimes\delta_\upsilon))$$

[illegible]

$$= \delta_{f(u)f(b\triangleright u)},\delta_{f(a\triangleright u)}\otimes f((a\triangleleft u)(b\triangleleft \upsilon))$$

$$= \delta_{u,b\triangleright u)}\delta_{a\triangleright u)},\otimes f((a\triangleleft \upsilon)(b\triangleleft \upsilon))$$

On the other hand,

$$= \delta_{u,b\triangleright u)}\delta_{f(a\triangleright(b\triangleright\upsilon)}\otimes f(a\triangleleft u))f(b\triangleleft u))(b\triangleleft u))$$

$$= \delta_{u,b\triangleright u)}\delta_{f(ab\triangleright u)}\otimes f(ab\triangleleft u)$$

We have utilized here the fact that f is an isomorphism and put $u = b\triangleright\upsilon$ to avoid having a zero answer. Next we check the effect of f on the unit, i.e., we want to show that $f(1_{\mathcal{H}})=1_{\mathcal{H}^*}$.So

$$\tilde{f}(1_{\mathcal{H}})=\tilde{f}\left(\sum_u e\otimes\delta_u\right)=\sum_u \delta_{f(u)}\otimes f(e\triangleleft u) = \ldots\ \otimes e = 1_{H^*}$$

where $= f(u)$ is an element of H, as required.

Now, the question arises "does the same result hold for the coalgebra". The answer is in negative as the counit property is not applicable unless we assume that our semigroup H posses, at least, the left inverse property as we see in the following lemma.

Lemma 3.3

Let X = GH be factorization of a group X into a subgroup G and a subsemigroup with identity and left inverse property H. Then for the coalgebra $\mathcal{H}$ = kH k(G) ,where k(G) is the algebra of function on G and kH is the semigroup algebra of H, there is a coalgebra homomorphism: $\mathcal{H} \to \mathcal{H}^*$ which sends basis elements to basis elements can be constructed from a factor-reversing isomorphism of X = GH.

Proof

We suppose that is a semigroup isomorphism and we consider the same linear map : $\mathcal{H} \to \mathcal{H}^*$defined in the proof of lemma 3.2 by

...(6)

where a∈H and u∈G. To check that satisfies the conditions to be a coalgebra homomorphism, we start by showing that Δf(a⊗δ_μ) = (⊗)Δ(a⊗δυ) as follows:

On the other hand,

$$= \sum_{xy=u} \tilde{f}(a \otimes \delta_x) \otimes \tilde{f}(a \triangleleft x) \otimes \delta_y)$$

$$= \sum_{xy=u} \delta_{f(a \triangleright x)} \otimes f(a \triangleleft x) \otimes \delta_{f((a \triangleleft x) \triangleright y)} \otimes f(a \triangleleft x) \triangleleft y)$$

$$= \sum_{xy=u} \delta_{f(a \triangleright x)} \otimes f(a \triangleleft x) \otimes \delta_{f((a \triangleleft x) \triangleright y)} \otimes f(a \triangleleft xy)$$

Putting m = f(a ▷ x) and n = f((a x) y) yields

mn = f(a x)f((a x) y)

= f((a x)((a x) y))

= f(a (xy)) = f(a u).

We have used the assumption that f is a semigroup homomorphism. Also, we get

n f(a u) = f((a x) y) f(a u)

= f((a x)$^{-1}$(a (xy))) f(a u)

= f((a x)$^{-1}$(a u)) f(a u)

= f(a x)$^{-1}$ (f(a u) f(a u))

= f(a x)$^{-1}$ f(a)

= f(a x)$^{-1}$ (f(a x) f(a x))

= f((a x)$^{-1}$(a x))f(a x)

= f(e)f(a x)

= f(a x),

as required. Next we check the effect of on the counit i.e., we want to prove that ∈H* (a⊗δ_μ) = ∈$\mathcal{H}$* (a⊗δ_μ) which we do as follows:

$$\triangleleft$$

$$\in_{H^*}\tilde{f}(a \otimes \delta_u) = \in_{H^*}(\delta_{f(a \triangleright u)} \otimes f(a \triangleleft u))$$

$$= \delta_{f(a \triangleright u),e}$$

$$= \delta_{u,e} = \in_H (a \otimes \delta_u).$$

To have a non-zero answer we have put f(a u) = e which implies that a u = e as f is an isomorphism. Applying a^L to both sides gives u = e.

Theorem 3.4

Let X = GH be factorization of a group X into a subgroup G and a subsemigroup with identity and left inverse property H. Then for the Hopf algebra $\mathcal{H}$ = kH k(G), where k(G) is the algebra of function on G and kH is the semigroup algebra of H, there is a Hopf algebra isomorphism: $\mathcal{H} \to \mathcal{H}^*$ which sends basis elements to basis elements can be constructed from a factor-reversing isomorphism of X = GH.

Proof

We Suppose that is a semigroup isomorphism and we consider the same linear map : $\mathcal{H} \to \mathcal{H}^*$ defined in the proof of lemma 3.2 by

$$\tilde{f}(a\otimes\delta_u)=d_{f(a\triangleright u)}\otimes f(a\triangleleft u) \quad ...(7)$$

where $a\in H$ and $u\in G$. The conditions for f to be an algebra and a coalgebra isomorphism follow from lemmas 3.2 and 3.3. To prove that ef is a Hopf algebra isomorphism, we need to check the antipode property and the inevitability of . First, we need the following calculations:

$(au)^L = ((a\quad u)(a\quad u))^L$

$u^L a^L = u^{-1}a^L = (a\quad u)^L(a\quad u)^L = (a\quad u)^L(a\quad u)^{-1}$

$\quad = ((a\quad u)^L\quad (a\quad u)^{-1})((a\quad u)^L\quad (a\quad u)^{-1})$.

By the uniqueness of factorization, we get

$u^L = u^{-1} = (a\quad u)^L\quad (a\quad u)^{-1}$ and

$a^L = (a\quad u)^L\quad (a\quad u)^{-1})$ ...(8)

Due to the fact that f is a semigroup isomorphism, we get

$f(u^{-1}) = f(u^L) = (f(u))^L = f((a\quad u)^L\quad (a\quad u)^{-1})$ ...(9)

$f(a^L) = (f(a))^L = (f(a))^{-1} = f((a\quad u)^L\quad (a\quad u)^{-1})$ ...(10)

Now to show that the antipode S is preserved under , i.e., S $(a\otimes\delta_u)$ = S $(a\otimes\delta_u)$, we do the following

$$\begin{aligned}\tilde{f}S(a\otimes\delta_u)&=\tilde{f}S(a\otimes\delta_u)\\&=\tilde{f}(a\triangleleft u)^L\otimes\delta_{(a\triangleright u)^{-1}})\\&=\delta_{f(((a\triangleleft u)^L\triangleright(a\triangleright u)^{-1})}\otimes f((a\triangleleft u)^L\triangleleft(a\triangleright u)^{-1})\\&=\delta_{(f(u))^L}\otimes f(a)^{-1}\end{aligned}$$

On the other hand,

$$\begin{aligned}S\tilde{f}(a\otimes\delta_u)&=S(\tilde{f}(a\otimes\delta_u))\\&=S(\delta_{f(a\triangleright u)^L}\otimes f(a\triangleleft u))\\&=\delta_{(f(a\triangleright u)\triangleleft f(a\triangleleft u))^L}\otimes(f(a\triangleright u)\triangleright f(a\triangleleft u))^{-1}\\&=\delta_{(f(u))^L}\otimes f(a)^{-1}\end{aligned}$$

as required. Finally, to see that : $\mathcal{H}^*\to H$ is invertible, we define $^{-1}$: $\mathcal{H}^*\to\mathcal{H}$ by

$$\tilde{f}^{-1}(\delta_a\otimes u)=f^{-1}(a\triangleright u)\otimes\delta_{f^{-1}(a\triangleleft u)} \quad ...(11)$$

and show that : $(\delta_a\otimes u)$ = $^{-1}$ $(a\otimes\delta_u)$ = $\mathrm{id}(a\otimes\delta_u)$ where id is the identity map, as follows

$$\begin{aligned}\bar{f}\bar{f}^{-1}(\delta_a\otimes u)&=\bar{f}(\bar{f}^{-1}(\delta_a\otimes u))\\&=\bar{f}(\bar{f}^{-1}(a\triangleright u)\otimes\delta_{f^{-1}(a\triangleleft u)})\\&=\delta_{f(f^{-1}(a\triangleright u)\triangleright f^{-1}(a\triangleleft u))}\otimes f(f^{-1}(a\triangleright u)\triangleleft f^{-1}(a\triangleleft u))\\&=\delta_{f(f^{-1}(a))}\otimes ff^{-1}(u)\\&=\delta_a\otimes u\end{aligned}$$

The third equality is due to the identities $f(a)= f(a\quad u)\quad f(a\quad u)$ and $f(u) = f(a\quad u)\quad f(a\quad u)$ with the fact that f is an isomorphism. Also we have

$$\begin{aligned}\bar{f}^{-1}\bar{f}(a\otimes\delta_u)&=\bar{f}^{-1}(\bar{f}(a\otimes\delta_u))\\&=\bar{f}^{-1}(\delta_{f(a\triangleright u)}\otimes f(a\triangleleft u))\\&=f^{-1}(f(a\triangleright u)\triangleright f(a\triangleleft u))\otimes\delta_{f^{-1}(f(a\triangleright u)\triangleleft f(a\triangleleft u))}\\&=f^{-1}(f(a))\otimes\delta_{f^{-1}(f(u))}\\&=\alpha\otimes\delta_u,\end{aligned}$$

as required. Therefore, f is a Hopf algebra isomorphism.

Following theorem reveals that the converse of Theorem 3.4 is also true.

Theorem 3.5

Let X = GH be factorization of a group X into a subgroup G and a subsemigroup with identity and left inverse property H. Then the factor-reversing isomorphisms of X = GH give rise to Hopf algebra self-duality pairings <,> : $H\otimes H\to k$ on the Hopf algebra $\mathcal{H}$ = kH   k(G) where k(G) is the Hopf algebra of function on G and kH is the semigroup Hopf algebra of H. The corresponding pairing is given by

$$\langle a\otimes\delta_u, b\otimes\delta_u\rangle=\delta_{\alpha,f(b\triangleright u)}\delta_{u,f(b\triangleleft u)}$$

Proof

Assume that $^{-1}$ is a Hopf algebra isomorphism which sends basis elements to basis elements of our two Hopf algebras, and we want to prove that we can induce a group isomorphism f^{-1} from f^{-1}. We start with functions h : H × G→H and g : H × G→G given by

$$\tilde{f}(a\otimes\delta_u)=\delta_{h(a,u)}\otimes g(a,u) \qquad ...(12)$$

As f is an algebra isomorphism, it preserves the unit and the product. Starting with the unit, we get

$$\bar{f}(1_H)=\bar{f}\left(\sum_u e\otimes\delta_u\right)=\sum_u\bar{f}(e\otimes\delta_u)=\sum_{h(e,u)}\delta_{h(e,u)}\otimes g(e,u)$$

...(13)

but, since is an algebra isomorphism we have

$$\tilde{f}(1_H)=1_{H^*}=\sum_a\delta_a\otimes e, \qquad ...(14)$$

for some s∈H. Comparing equations (13) and (14) gives

$$g(e,u=e) \qquad ...(15)$$

Now, for the product we have

$$\begin{aligned}\tilde{f}((a\otimes\delta_u)(b\otimes\delta_u))&=\tilde{f}(\delta_{u,b\triangleright\upsilon};(ab\otimes\delta_\upsilon))\\&=\delta_{u,b\triangleright\upsilon};\tilde{f}(ab\otimes\delta_\upsilon)\\&=\delta_{u,b\triangleright\upsilon};(\delta_{h(ab,\upsilon)}\otimes g(ab,\upsilon))\end{aligned}$$

On the other hand,

$$\begin{aligned}\bar{f}(a\otimes\delta_u)\bar{f}(b\otimes\delta_u)&=(\delta_{h(a,u)}\otimes g(a,u))(\delta_{h(b,u)}\otimes g(b,u))\\&=(\delta_{h(a,u)\triangleleft g(a,u),h(b,u)}(\delta_{h(a,u)}\otimes g(a,u)g(b,u))\\&=(\delta_{h(a,u)\triangleleft g(a,u),h(b,u)}(\delta_{h(a,u)}\otimes g(a,u)g(b,u))\end{aligned}$$

...(17)

To have non-zero answer we should have u = b υ and

$$h(b,\upsilon)=h(a,u)\triangleleft g(a,u) \qquad ...(18)$$

Equations (16)and (17) imply that for all a, b∈H and u, v ∈ G, the following equalities are satisfied:

$$g(ab,\upsilon)=g(a,u)g(b,\upsilon) \qquad ...(19)$$

$$h(ab,\upsilon)=h(a,u) \qquad ...(20)$$

Note that if we put υ = e in (19) and substitute u = b v, we get

$$g(ab,e)=g(a,e)g(b,e)$$

...(21)

Next, as is a coalgebra isomorphism, it preserves the counit and the coproduct. So we start with the counit as follows

$$\epsilon_{H^*}\tilde{f}(a\otimes\delta_u)=\epsilon_{H^*}(\delta_{h(a,u)}\otimes g(a,u))=\delta_{h(a,u),e}, \qquad ...(22)$$

but as is a coalgebra isomorphism, we have

$$\epsilon_{H^*}\tilde{f}(a\otimes\delta u)=\epsilon_H(a\otimes\delta_u)=\delta_{u,e} \qquad ...(23)$$

Combining (22) and (23)and putting u = e, to have a non-zero solution, imply

$$h(a,e)=e \qquad ...(24)$$

▷Now we calculate the coproduct under f to have

$$\Delta\tilde{f}(a\otimes\delta_u)=\Delta(\delta_{h(a,u)}\otimes g(a,u))$$

$$=\sum_{mn=h(a,u)}\delta_m\otimes(n\triangleright g(a,u))\otimes\delta_n\otimes g(a,u) \qquad ...(25)$$

On the other hand, since is a colagebra isomorphism, we have

$$\begin{aligned}\Delta\bar{f}(a\otimes\delta_u)&=(\bar{f}\otimes\bar{f})\Delta(a\otimes\delta_u)\\&=(\bar{f}\otimes\bar{f})\left(\sum_{xy=u}(a\otimes\delta_x)\otimes((a\triangleleft x)\otimes\delta_y)\right)\\&=\sum_{xy=u}\bar{f}(a\otimes\delta_x)\otimes\bar{f}((a\triangleleft x)\otimes d_y)\\&=\sum_{xy=u}\delta_{h(a,x)}\otimes g_{(a,x)}\otimes\delta_{h(a\triangleleft x,y)}\otimes g(a\triangleleft x,y)\end{aligned}$$

...(26)

From equations (25) and (26), we get

$$h(a,u)=mn=h(a,x)h(a\triangleleft x,y)=h(a,xy)$$

Putting a = e gives

$$h(a,u)=h(e,x)h(e\triangleleft x,y)=h(e,x)h(e,y)=h(e,xy)$$

...(27)

We also have, from the coproduct formula, that n g(a, u) = g(a, x) where n = h(a x, y) and xy = u. Putting x = e gives

$$h(a \triangleleft e,\ y)\ \triangleright\ g(a,\ u)\ =\ g(a,\ e),$$

or

$$h(a,\ y)\ \triangleright\ g(a,\ u)\ =\ g(a,\ e) \quad \text{...(28)}$$

Since we have xy = u, putting x = e gives y = u. Thus equation (28) can be rewritten as

$$h(a,\ u)\ \triangleright\ g(a,\ u)\ =\ g(a,\ e) \quad \text{...(29)}$$

From (18) with v = b^L u and b = e we get

$$h(a,\ u)\ \triangleright\ g(a,\ u)\ =\ h(e,\ u) \quad \text{...(30)}$$

Combining equations (29) and (30) gives

...(31)

Putting a = e in (20) yields

$$h(b,\ v)\ =\ h(e,\ u)$$

Knowing that u = b ▷ v implies

$$h(b,\ v)\ =\ h(e,\ b\ \triangleright\ v) \quad \text{...(32)}$$

Also, from the coproduct formula, we get g(a x, y) = g(a, u) with u = xy, i.e., g(a x,y) = g(a, xy). Putting y = e gives

$$g(a\ \triangleleft\ x,\ e)\ =\ g(a,\ x)$$

combining equations (32) and (31) gives

$$h(e, a \triangleright u)g(a \triangleleft u, e) = h(a,u)g(a,u) = g(a,e)h(e,u). \quad \text{...(33)}$$

Equations (15), (24), (21), (27), and (33) provide the needed conditions ensuring that the map f^{-1} : X→X defined by

$$f^{-1}(au) = g(a,\ e)h(e,\ u).$$

is a group homomorphism. It can be noted that our Hopf algebra map f^{-1} is certainly that one obtained by $^{-1}$, which is well defied due to G ηH = {e}. Since f^{-1} is a Hopf algebra isomorphism, it is invertible. So if we put

$$\bowtie h(a,u)\ g(a,u)\ \tilde{f}(\delta_{\alpha} \otimes u)\ =\ h(a,\ u)\ \ \delta_{g}(a,u).\ e)\ h(e,u)$$

it can be easily shown that f is obtained by the group isomorphism by using a similar technique.

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