INTRODUCTION

The fact that a Hopf algebra $H = kM\triangleright\triangleleft k(G)$ can be constructed for every factorization $X = GM$ of a group into two subgroups $G$ and $M$, is well known. This bicrossproduct construction is one of the sources of true non-commutative and non-cocommutative Hopf algebras$^{7-10}$. These bicrossproduct Hopf algebras have been studied intensively$^{1-6}$. In$^{3}$, Beggs has considered more general factorizations of groups, and the corresponding algebras. More specifically, it was shown that it is possible to construct a non-trivially associated tensor category $C$ from data which is a choice of left coset representatives $M$ for a subgroup $G$ of a finite group $X$. The objects of this category are the right representations of $G$ that possess $M$-grading. The group action and the grading in the definition of $C$ were combined by considering a single object $A$ spanned by a basis $\delta_s \otimes u$ for $s \in M$ and $u \in G$. This object, $A$, was shown to be an algebra in $C$ under certain conditions.

In this paper, we show that for the algebras $A$ and $A$, the map $\theta: A \to A$ defined by $\theta((\delta_s \otimes u)) = \delta_{s \triangleright u} (s \triangleright u)^{-1}u$, where $(\delta_s \otimes u) \in A$ and $A = \langle \delta_s \otimes u \rangle$, is a morphism in the category $C$. Moreover, it is shown that the morphism $\theta: A \to A$ is an algebra map.

Key words: Finite group factorization, Algebras, Bicrossproduct Algebras.

ABSTRACT

In this paper, we show that for the algebras $A$ and $A$, the map $\theta: A \to A$ defined by $\theta((\delta_s \otimes u)) = \delta_{s \triangleright u} (s \triangleright u)^{-1}u$, where $(\delta_s \otimes u) \in A$ and $A = \langle \delta_s \otimes u \rangle$, is a morphism in the category $C$. Moreover, it is shown that the morphism $\theta: A \to A$ is an algebra map.

We use the same formulas and ideas given in$^{3}$, which is based on the papers$^{4,5}$. Throughout the paper we assume that all mentioned groups are finite, and that all vector spaces are finite dimensional over a field $k$.

Preliminaries

In order to make the paper self contained, we include the following definitions and results of$^{3}$.

Definition

For a group $X$, consider the factorization $X = GM$ where $G$ is a subgroup of $X$ and $M \subseteq X$ is a set of left coset representatives for a subgroup $G$ of a finite group $X$. The objects of this category are the right representations of $G$ that possess $M$-grading. The group action and the grading in the definition of $C$ were combined by considering a single object $A$ spanned by a basis $\delta_s \otimes u$ for $s \in M$ and $u \in G$. This object, $A$, was shown to be an algebra in $C$ under certain conditions.

In this paper, we show that for the algebras $A$ and $A$, the map $\theta: A \to A$ defined by $\theta((\delta_s \otimes u)) = \delta_{s \triangleright u} (s \triangleright u)^{-1}u$, where $(\delta_s \otimes u) \in A$ and $A = \langle \delta_s \otimes u \rangle$, is a morphism in the category $C$. Moreover, it is shown that the morphism $\theta: A \to A$ is an algebra map.

$\triangleright (t \triangleright u) = \tau(s,t)((s,t) \triangleright u) \tau(s \triangleleft (t \triangleright u), (t \triangleright u))^{-1}$

We use the same formulas and ideas given in$^{3}$, which is based on the papers$^{4,5}$. Throughout the paper we assume that all mentioned groups are finite, and that all vector spaces are finite dimensional over a field $k$.

Definition

For a group $X$, consider the factorization $X = GM$ where $G$ is a subgroup of $X$ and $M \subseteq X$ is a set of left coset representatives. For $x \in X$ the factorization $\chi = us$ for $u \in G$ and $s \in M$ is unique. Define $\tau(s,t) \in G$ and $s \cdot t \in M$ as well as the functions $\triangleright: M \times G \to G$ and $\triangleleft: M \times G \to M$ by the unique factorizations in $X$: $st = \tau(s,t)(s \cdot t)$ and $su = \tau(s,t)(s \cdot t)$ for $s, s \triangleleft u \in M$ and $u, s \triangleright u \in G$.

For $t, s, p \in M$ and $u, v \in G$, the following identities hold:

$\triangleright (t \triangleright u) = \triangleright (s \triangleright u) = (s \triangleright (t \triangleright u)) \triangleright u = \tau(s,t)(s \triangleright u) \triangleright u = (s \triangleright u)(s \triangleright v)$,
The category $C$ is defined to be a category of finite dimensional vector spaces over a field $k$, whose objects are right representations of the group $G$ and have $G\backslash \mathcal{X}$-gradings, i.e., an object $V$ can be written as $\bigoplus_{s \in G\backslash \mathcal{X}} V_s$. The action for the representation is written as $\ell : V \times G \to V$. In addition, it is supposed that the action and the grading satisfy the compatibility condition, i.e., $\langle \xi \cdot u \rangle = \langle \xi \rangle \cdot u \cdot \ell$.

The morphisms in the category $C$ is defined to be linear maps which preserve both the grading and the action, i.e., for a morphism $\varphi : V \to W$ we have $\langle \varphi(\xi \cdot u) \rangle = \langle \varphi(\xi) \rangle \cdot \varphi(\varphi^{-1}(\xi)) \cdot u$ for all $\xi \in V$ and $u \in G$.

$C$ is a tensor category with actions and gradings given by $\otimes$. There is an associator $\Phi_{UW} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ given by

$$\Phi((\xi \otimes \eta) \otimes \zeta) = \xi \otimes (\eta \cdot \zeta).$$

The dual object of $V \in C$ is of the form $V^* = \bigoplus_{s \in \mathcal{M}} V_s^*$, and we define $(a)^* = s^* \cdot a$ when $a \in V_s$. The evaluation map $ev : V^* \otimes V \to k$ is defined by $ev(a, \xi) = a(\xi) = (a(\langle \xi \rangle \cdot u) \xi \cdot u)$, or if we put $\eta = \xi \cdot u$ we get $a(\eta \cdot u^{-1}) = (a(\langle \eta \rangle \cdot u^{-1}))$. If this is rearranged to give $\eta = a(\langle \eta \rangle \cdot u^{-1}) \cdot (\eta \cdot u^{-1})$, then $\Phi(\xi \otimes \eta \otimes \zeta) = \xi \otimes \eta \otimes \zeta$.

The coevaluation map, which is a morphism in $C$ is defined by:

$$\text{coev}(1) = \sum_{\xi \text{ basis}} \xi \otimes (\langle \xi \rangle \cdot \overline{\xi}) \cdot (\langle \xi \rangle ^{-1} \otimes \overline{\xi}),$$

where $\overline{\xi}$ is a corresponding dual basis of $\xi \in V_s$, $\forall s \in \mathcal{M}$.

The dependence on the choice of representatives was explained as follows: For a given subgroup $G$ of a group $X$ we can choose different sets of representatives $\mathcal{M}$ and $\overline{\mathcal{M}}$ for the left cosets. These are related by an arbitrary function $\gamma : G \times G \to \{0, 1\}$, so that if $s \in \mathcal{M}$ then $\gamma([s])s \in \mathcal{M}$. We now have two sets of binary operations "cocycles" and "left "actions". We write these as $\cdot$ and $\cdot : G \times G \to G$ and $\cdot : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, respectively. The right action in both cases is the canonical right action by $G$ on $G \times G$. The following equations connect the different operation:

$$t \cdot u = \gamma(t)(t \cdot u), \quad s \cdot t = (s \cdot \gamma(t)) \cdot t,$$

$$\tau(s, t) = \gamma(s) \cdot \gamma(t) \cdot \tau(s \cdot \gamma(t), t) \gamma((s \cdot \gamma(t)) \cdot t)^{-1}$$

...(3)\]

Instead of representatives, we should have phrased the last equations in terms of cosets $[s] = Gs$.

The tensor structure given by $\mathcal{M}$ in the category $C$ shall be called $\otimes$ as usual and the tensor structure given by $\overline{\mathcal{M}}$ shall be called $\otimes$. The map $F_{vw} : V \otimes W \to V \otimes W$ is defined by $F(\xi \otimes \eta) = \xi \cdot \gamma(\eta)$, and this gives an equivalence of the two tensor product structures. To check that $F$ is a morphism in $C$ we note that $F$ preserve the $G \times G$ grade from the equation for the binary operation given in (3). For the $G$-action we use

$$F(\xi \otimes \eta) \cdot u = \xi \cdot \gamma(\eta) \cdot u = \xi \cdot \gamma(\eta) \cdot u \otimes \eta \cdot u = F(\xi \otimes \eta) \cdot u = F(\xi \otimes \eta) \cdot u$$

Now, we consider morphisms $\varphi : V \to W$ and $\chi : W \to V$ in $C$. It was shown that the following diagram commutes:
For the dual of an object $V$ in $C$, we use the assumption on the rigid tensor category structure, which are that the coset representatives include $e$ and have right inverse. As $X$ is a finite group, this implies that the left inverse map is 1-1, i.e., that if $s^t = t$ then $s = t$. The dual $V'$ is the dual vector space, but the $G\times X$ grading and $G$ action depend on $M$. Remember that if $\alpha \in V'$ annihilates all $V_p$ for $p \neq q \in G\times X$, then it has grade $q^t$, but the left inverse depends on the choice of $M$. We call $V'$ the dual for the coset representatives $M$, and use $\langle \cdot \rangle$ and $\cdot$ for the grading and the action on $V'$.

It was shown, in \(^3\), that the map $H_v : V' \rightarrow V'$ defined by $H_v(\alpha) = \alpha \cdot \gamma(\phi)^{-1}$ where $\alpha \in V'$ annihilates all $V_p$ for $p \neq q \in G\times X$, is a morphism in $C$. Moreover, it was proved that the following diagrams commute:

$$
\begin{array}{ccc}
V \otimes V & \xrightarrow{eval} & k \\
\downarrow F v v & & \downarrow \text{id} \\
V' \otimes V & \xrightarrow{eval} & k
\end{array}
$$

The algebra $A$ in the tensor category $C$ is constructed so that the group action and the grading in the definition of $C$ can be combined. Consider a single object $A$, a vector space spanned by a basis $\delta_s \otimes u$ for $s \in M$ and $u \in G$. For any object $V$ in $C$ define a map $\xi : \delta_s \otimes u \rightarrow V$ by $\xi \cdot \delta_s \otimes u = \delta_s \otimes \xi u$. This map is a morphism in $C$ only if $\langle \xi u \rangle = \langle \xi u \rangle$ i.e., $\langle \xi u \rangle = \langle \xi u \rangle u$ if $\langle \xi u \rangle = \langle \xi u \rangle$. If we put $a = \langle \xi u \rangle$, the action of $\alpha \in G$ is given by $\delta_s \otimes \alpha = \delta_s \otimes (a \cdot v)^{-1} \alpha$. 

[Diagram of tensor products and morphisms]
Different choices of coset representatives

In [3] it was stated that for a given subgroup \( G \) of a group \( X \), different sets of representatives \( M \) and \( \bar{M} \) for the left cosets can be chosen and these are related by an arbitrary function \( \gamma : G \times G \rightarrow G \) so that if \( s \in M \) then \( \gamma ([s]) \in \bar{M} \). Also the binary operations \( \cdot : G \times G \rightarrow G \) and \( \triangleright : G \times G \rightarrow G \) for \( M \) were shown to be the following:

\[
\begin{align*}
\tau s &= (s < \gamma(t), t), \\
\tau(s, t) &= \gamma(s)(s > \gamma(t)) \\
\tau(t > u) &= \gamma(t)(t > u)^{-1}
\end{align*}
\]

Here we prove that for the algebras \( A \) and \( A \), the map \( \theta : A \rightarrow A \) defined by \( \theta(s \otimes u) = \delta_s \otimes \gamma(u) \), where \( (s \otimes u) \in A \) and \( \gamma(u) \) is a morphism in the category \( \mathcal{C} \). In addition, we prove that the morphism \( \theta : A \rightarrow A \) is an algebra map.

Proposition

For the algebras \( \mathbb{A} \) and \( A \), the map \( \theta : \mathbb{A} \rightarrow A \) defined by

\[
\theta(s \otimes u) = \delta_s \otimes \gamma(u)
\]

where \( (s \otimes u) \in \mathbb{A} \) and \( \gamma(u) \) is a morphism in the category \( \mathcal{C} \).

Proof

We should first show that \( \theta(s \otimes u) \) preserves the grades. Let \( \mathfrak{a} = (s \otimes u) \) then \( \mathfrak{a} \) is defined by

\[
\mathfrak{a} = s < u = s < u,
\]

but also we have

\[
(s < \gamma(u)) \cdot \theta(s \otimes u) = (s < \gamma(u)) < \gamma(u)^{-1}
\]

\[
u = s < \gamma(u) \gamma(u)^{-1} u = s < u,
\]

so it preserves the grades. Now we need to check that it also preserves the actions, i.e.

\[
\theta(s \otimes u) = \delta_s \otimes \gamma(u)
\]

To calculate the left hand side we need to calculate the following:

\[
\delta \bigg|_{s \otimes (\otimes u)} \otimes \gamma \bigg|_{(\otimes u)^{-1}} \nu
\]

\[
\delta \big|_{s \otimes (\otimes u)} \otimes \gamma \big|_{(\otimes u)^{-1}} \gamma^{-1} \nu
\]

So

L.H.S.

\[
\delta \bigg|_{s \otimes (\otimes u)} \otimes \gamma \bigg|_{(\otimes u)^{-1}} \gamma^{-1} \nu
\]

Now we calculate the right hand side

R.H.S. =

\[
\theta(s \otimes u) < \nu = (s \otimes u) < \gamma^{-1}(\theta^{-1}) \nu
\]

which shows that \( \theta \) preserves the actions.

In [3], the morphism \( F_{\otimes w} : \mathcal{V} \otimes \mathcal{W} \rightarrow \mathcal{V} \otimes \mathcal{W} \) for \( V, W \in \mathcal{C} \) and \( \otimes \) is the tensor structure given by \( \mathcal{M} \), was defined by \( F(\xi \otimes \eta) = \xi \gamma((\otimes)) \otimes \eta \). This morphism will be used in the next proposition.

Proposition

For the algebras \( \mathbb{A} \) and \( A \), the morphism \( \theta : \mathbb{A} \rightarrow A \) is an algebra map. Note that we have to be consistent with the notation.
careful about just what this means, as there are two different tensor products. We mean that the following diagram commutes:

\[ \begin{array}{c}
\delta \otimes u \\
\delta \otimes v
\end{array} \]

Proof
For the elements \((\delta \otimes u)\) and \((\delta \otimes v)\) in the algebra \(A\) we have

\[ \mu((\delta \otimes u) \otimes (\delta \otimes v)) = \delta \otimes \delta \otimes \tau(a) \otimes \tau(b)^{-1} uv, \]

where

\[ a = (\delta \otimes u) \text{ and } b = (\delta \otimes v) \]

so

\[ \phi\left(\mu((\delta \otimes u) \otimes (\delta \otimes v))\right) = \delta \otimes \delta \otimes \tau(a) \otimes \tau(b)^{-1} \otimes \gamma(a^{-1}) \]

But we know that

\[ \tau(a, b) = \gamma(a) \tau(a < \gamma(b), b) \gamma((a < \gamma(b)), b)^{-1} \]

and

\[ a, b = (a < \gamma(b), b) \]

So

\[ \tau(a, b) = \gamma(a) \tau(a < \gamma(b), b) \gamma((a < \gamma(b)), b)^{-1} \]

Thus, one direction of the diagram is given by the following equation:

\[ \mu\left(F_{\otimes}\left(\phi\left(\mu((\delta \otimes u) \otimes (\delta \otimes v))\right)\right)\right) = \delta \otimes \delta \otimes \tau(a) \otimes \tau(b)^{-1} \otimes \gamma(a^{-1}) \]

Now to calculate the other direction of the diagram we do the following calculation:

\[ \phi\left(\mu((\delta \otimes u) \otimes (\delta \otimes v))\right) = \delta \otimes \delta \otimes \tau(a) \otimes \tau(b)^{-1} \otimes \gamma(a^{-1}) \]

Applying the map \(F_{\otimes}\) to the above equation gives

\[ F_{\otimes}\left(\phi\left(\mu((\delta \otimes u) \otimes (\delta \otimes v))\right)\right) = \delta \otimes \delta \otimes \tau(a) \otimes \gamma(a^{-1}) \]

Therefore, the other direction of the diagram is given by the following equation:

\[ \mu\left(F_{\otimes}\left(\phi\left(\mu((\delta \otimes u) \otimes (\delta \otimes v))\right)\right)\right) = \delta \otimes \delta \otimes \tau(a) \otimes \gamma(a^{-1}) \]

which is the same as the first direction and this completes the proof.

REFERENCES


