Some study of a cauchy problem in parabolic integrodifferential equations class

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ABSTRACT

We consider the parabolic integrodifferential equations of a form given below. We establish local existence and uniqueness and prove the convergence in \( L_2(E) \) to the solution \( u_t \) of the Cauchy problem.

Key words: Cauchy, parabolic, integrodifferential equations, operator.

INTRODUCTION

Let's have the parabolic integrodifferential equations of the form

\[
\frac{du}{dt} - \sum_{|\alpha|<2m} a_\alpha(t) \partial^\alpha u + \int_0^t \sum_{|\alpha|<2m} k_\alpha(t, \theta) \partial^\alpha u d\theta + \sum_{|\alpha|<2m} \alpha(x,t) \partial^\alpha u_t + f_t,
\]

where the partial differential operator \( \sum_{|\alpha|<2m} a_\alpha(t) \partial^\alpha \) is uniformly elliptic, \( \int_0^t \sum_{|\alpha|<2m} k_\alpha(t, \theta) \partial^\alpha u d\theta \) is a family of linear bounded operators defined on the space of all square integrable functions \( L_2(E) \) and \( E \) is the n-dimensional Euclidean space.

We consider integrodifferential equations of the form:

\[
\frac{du}{dt} = L u + f_t, \quad t > 0,
\]

Where, \( \alpha = (a_1, \ldots, a_n) \) is an n-dimensional multi index, \( |\alpha| = a_1 + \cdots + a_n \).
Notice that if \( 4m > n, u \in W^{n,2}(E^n), \frac{\partial u}{\partial t} \in W^{n,2}(E^n), \) for each \( t \in (0, T], \) then the partial derivative \( \frac{\partial u}{\partial t} \) exists in the usual sense \( E^n \times (0, T] \) and \( \frac{\partial u}{\partial t} \) in fact according to the embedding theorem \([3], [8],\) we have

\[
\max_{\partial \Omega} \left| \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} \right| \leq C \left\| \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} \right\|_{W^{2n}(E^n)}
\]

Where \( a_\mu = \omega_\mu + \omega_{\mu} . c \) is a positive constant.

Let us suppose that the following assumptions are satisfied;

(a) The coefficients \( a_\mu, |\alpha| = 2m \) are real functions of \( t, \) defined on \([0, T]\) and having continuous derivatives \( \frac{\partial a_\mu}{\partial t} \) on \([0, T]\)

(b) The differential operator \( \sum_{\mu=1}^{\lambda} a_\mu(x, t) L^\mu \) is uniformly elliptic on \([0, T].\)

(c) The kernels \( \{K_\mu(\theta, \phi) \leq 2m, \theta, \phi \in [0, T] \} \) are linear bounded operators acting on \( L^2(E^n) \) into itself. It is assumed that these operators are continuous in \( (t, \theta), t, \theta \in [0, T]. \)

Furthermore it is assumed that the (abstract) partial derivatives \( \frac{\partial K_\mu}{\partial \theta} \) exist for all \( |\alpha| = 2m \) and represent linear bounded operators on \( L^2(E^n) which are continuous in \( (t, \theta), t, \theta \in [0, T]. \)

(d) The coefficients \( a_\mu, |\alpha| < 2m \) are real functions, which are continuous and bounded on \( [0, T]. \)

(e) \( f \) is a map from \([0, T] \) into \( L^2(E^n), \) which is continuous in \( t \) with respect to the norm in \( L^2(E^n). \)

(f) All the coefficients \( a_\mu \), for \( |\alpha| < 2m \) have continuous bounded partial derivatives \( D_\alpha a_\mu \) on \( E^n \times [0, T]; r = 1, \ldots, n. \)

(h) The range of the operators \( K_\mu(t, \theta)(|\alpha| \leq 2m, t, \theta \in [0, T]) \) is the space \( w^{n-2}(E^n) \). We assume that all the operators \( D_\alpha K_\mu(t, \theta) \) are bounded, and that \( D_\alpha K_\mu(t, \theta) \) exist for all \( |\alpha| = 2m. \) The operators \( D_\alpha K_\mu(t, \theta) \) are supposed to be bounded and continuous in \( (t, \theta) \) for all \( |\alpha| = 2m. \) It is supposed also that \( D_\alpha k_\mu(t, \theta) \) are continuous in \( (t, \theta) \)

**Proposition 1.**

Under conditions (a), (b), ..., (e) if there is at least one solution in the class \( S \) of the Cauchy problem (*), (**) then this solution is the unique such solution.

**Proof.** If \( u \in W^{n,2}(E^n), |\alpha| = 2m \) then we have the following representation:

\[
D^\alpha g = (-1)^m R^\alpha \nabla^{2m} g, \quad \ldots (1)
\]

Where \( R = R^m \cdots R^1 R \) is the singular integral operator defined to be the

\[
\lim_{k \to 0} \left\| R_j(h) - R_j \right\| = 0,
\]

\[
R_j(h) g = -i \pi^{(n+1)/2} \Gamma \left( \frac{n+1}{2} \right) \int_{|x-y|=1} \frac{x_j - y_j}{|x-y|^n} dy,
\]

\[
|x|^2 = x_1^2 + \ldots + x_n^2, \quad i = \sqrt{-1}, \quad V^2 = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}
\]

and

\[
\Gamma(\alpha) = \int_0^\infty e^{-t^{\alpha-1}} dt.
\]

Notice that \( r, j = 1, \ldots, n \) are bounded operators from \( L^2(E^n) \) into itself, [1],[2].

Let \( H_1(t) \) be an operator defined by

\[
H_1(t) = \sum_{|\alpha| \leq 2m} a_\alpha(t) R^\alpha.
\]
According to assumption, the operator $H_1(t)$ has a bounded inverse $H_1^{-1}(t)$ defined on for every $t \in [0, T]$.

Set $H_2(t, \theta) = \sum_{|k| \leq 2m} K_a(t, \theta) R^a$ and

$H_3(t, \theta) = H_2(t, \theta) H_1^{-1}(\theta)$. 

Using (1), then equation (*) can be written in the form;
$$\frac{du}{dt} + \sum_{|k| \leq 2m} a_k(t) D^a u + \int_{[0, t]} K_a(t, \theta) D^a u d\theta = f \quad (2)$$

To prove the uniqueness of the considered Cauchy problem, we set $f = 0$ for all $(x, t) \in \mathcal{B}_B \times [0, T]$. 

Now set
$$\frac{du}{dt} - \sum_{|k| \leq 2m} a_k(t) D^a u = u.$$ 

Then according to assumptions (1) and (2), we can write
$$u_t(x) = \int_0^t \int_{\mathcal{B}_B} G(x - y, t, \eta) V_\eta(\eta) dy d\eta \quad (3)$$

Where $G$ is the fundamental solution of the Cauchy problem for the parabolic equation

$$\frac{du}{dt} = \sum_{|k| \leq 2m} a_k D^a u.$$ 

Let $\{ U(t, \theta); t \in [0, T] \}$ be the family of bounded operators defined by

$$U(t, \theta) = \int_{\mathcal{B}_B} G(x - y, t, \theta) V_\eta(\eta) dy.$$ 

Consequently (3) can be written in the form:
$$u_t = \int_0^t U(t, \theta) V_\theta d\theta \quad (4)$$

According to the well-known properties of the fundamental solution $G$, [4], [5], we can see that
$$\| U(t, \theta) g \| \leq C \| g \|, g \in L_2(B_B), \quad (5)$$

$$D^a U(t, \theta) g \| \leq \frac{C}{(t - \theta)^\gamma} \| g \|, \quad (6)$$

For $|a| < 2m, t > \theta, g \in L_2(B_B)$, where $C$ is a positive constant and is a constant satisfying $0 < \gamma < 1$. Substituting from (4) into (2), we get

$$V_\gamma = \int_0^t H_3(t, \theta) V_\eta d\theta - \int_0^t \sum_{|k| \leq 2m} a_k(t) D^a U(t, \theta) V_\eta d\theta + \sum_{|k| \leq 2m} a_k(t) \int_0^t D^a U(t, \theta) V_\eta d\theta$$

$$+ \sum_{|k| \leq 2m} a_k(t) \int_0^t D^a U(t, \theta) V_\eta d\theta \quad (7)$$

Using (5) and (6), we get from (7), the following estimation;
$$\| V_\gamma \| \leq C \int_0^t \| V_\eta \| d\eta \quad (8)$$

(To obtain (7) and (8), we already used conditions (c) and (d) where $C$ is a positive constant. Thus (4) and (8) lead immediately to the fact that $u_t(x) = 0$ on $B \times [0, T]$.

Proposition 2

Under the condition (a), ..., (h) the solution of the Cauchy problem (*) , (**) exists in the class $S$.

Proof

Using the conditions from (a) to (e), we obtain

$$V_\gamma = \int_0^t H_3(t, \theta) V_\eta d\theta - \int_0^t \sum_{|k| \leq 2m} a_k(t) D^a U(t, \theta) V_\eta d\theta + \sum_{|k| \leq 2m} a_k(t) \int_0^t D^a U(t, \theta) V_\eta d\theta$$

$$H_3(t, \theta) \sum_{|k| \leq 2m} a_k(t) D^a U(t, \theta) V_\eta d\theta$$

$$+ \sum_{|k| \leq 2m} a_k(t) \int_0^t D^a U(t, \theta) V_\eta d\theta \quad (9)$$

(To obtain (9), we already used conditions (c) and (d) where $C$ is a positive constant. Thus (4) and (9) lead immediately to the fact that $u_t(x) = 0$ on $B \times [0, T]$.

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(To obtain (9), we already used conditions (c) and (d) where $C$ is a positive constant. Thus (4) and (9) lead immediately to the fact that $u_t(x) = 0$ on $B \times [0, T]$.
According to (5) and (6), the Volterra integral equation (9) has a unique solution \( v \) in which satisfies:

\[
\int_0^t U(t, \theta) V_\theta d\theta
\]

where \( c \) is a positive constant.

This means that under conditions from (a) to (e), we can obtain the so called mild solution [6] of the Cauchy problem (*), (**). This solution is represented by

\[
u_t = \int_0^t U(t, \theta) V_\theta d\theta
\]

To prove that \( u_t \in W^{2m}_{L(E)} \), for all \( t \in (0, T] \), we apply formally the differential operator to both sides of the integral equation (9), then we get:

\[
D_t v_t = -\int_0^t D_t H_2(t, \theta) V_\theta d\theta
\]

Let us consider as the unknown element in the integral equation (10). Under the assumptions (a),... (h) this integral equation can be solved for \( D_t v_t \). Thus \( D_t v_t \in L(L(E)) \) for \( t \in (0, T] \). Now we have

\[
D_t u_t = \int_0^t D_t U(t, \theta) V_\theta d\theta = \int_0^t U(t, \theta) D_t V_\theta d\theta
\]

Using (11), we can write

\[
D^\alpha u_t = \int_0^t D^\beta U(t, \theta) D_t V_\theta d\theta,
\]

where \( |\alpha| = 2m, |\beta| = 2m - 1 \).

**Proposition 3**

Let \( \{f_t\} \) be a sequence of functions which belong to \( L(L(E)) \) for every \( t \in (0, T] \) and continuous in \( t \). Suppose

\[
\lim_{\gamma \to 0} \sup_{t} \| f^\gamma_t - f_t \| = 0
\]

where \( f_t \) is continuous in \( t \in [0, T] \) if is a sequence of functions of the class \( S \), which are solutions of the Cauchy problem

\[
u_t = \int_0^t U(t, \theta) V_\theta d\theta
\]

\[
\frac{du_t^\gamma}{dt} = Lu_t^\gamma + f_t^\gamma, u_t^\gamma = 0,
\]

then \( \{u_t^\gamma\} \) converges in \( L(L(E)) \) to the solution of the Cauchy problem (*), (**).

**Proof.** Set

\[
u_t = \int_0^t U(t, \theta) V_\theta d\theta
\]

We find

\[
\| v_t^\gamma - v_t \| \leq \sup_{t} \| f_t^\gamma - f_t \| e^\alpha.
\]
REFERENCES