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Some study of a cauchy problem in parabolic integrodifferential equations class

A. S. AL-FHAID

Department of Mathematics, College of Sciences, P.O.Box 80203
King Abdulaziz University, Jeddah - 21589 (Saudi Arabia).

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ABSTRACT

We consider the parabolic integrodifferential equations of a form given below. We establish local existence and uniqueness and prove the convergence in $L_2(E_n)$ to the solution U_t of the Cauchy problem.

Key words: Cauchy, parabolic, integrodiffrential equationss, operator.

INTRODUCTION

Let's have the parabolic integrodifferential equations of the form

$$\frac{du_t}{dt} - \sum_{|\alpha|=2m} a_\alpha(t) D^\alpha u_t = \int_0^t \sum_{|\alpha|\le 2m} k_\alpha(t,\theta) D^\alpha u_\theta d\theta$$

$$+ \sum_{|\alpha|<2m} a_\alpha(x,t) D^\alpha u_t + f_t,$$

where the partial differential operator $\sum a_\alpha(t) D^\alpha$ is uniformly elliptic, $\{k_\alpha(t,\theta; |\alpha| \le 2m, t, \theta \in [0,T]\}$ is a family of linear bounded operators defined on the space of all square integrable functions $L_2(E_n)$ and E_n is the n- dimensional Euclidean space.

We consider integrodifferential equations of the form;

$$\frac{du_t}{dt} = Lu_t + f_t, t > 0,$$

Where,

$\alpha = (\alpha_1,..,\alpha_n)$ is an n-dimensional multi index, $|\alpha| = \alpha_1 + ... + \alpha_n$,

$$D^\alpha = D^{\alpha 1} ... D^{\alpha n}_n, D_r = \frac{\partial}{\partial x_r},$$

$r = 1,\ldots,n, x = (x_1,\ldots x_n)$ is an element of the n-dimensional Euclidean space E_n. Let $L_2(E_n)$ be the space of all square integrable function on E_n and $W^m(E_n)$ the Sobolev space, [the space of all functions $g \in L_2(E_n)$ such that he distributional derivative $D^\alpha g$ with $|\alpha| \le m$ all belong to $L_2(E_n)$, [7], [8]].

We assume that u_t satisfy the Cauchy condition;

$$u_0(x) = g(x), g \in W^{2m}(E_n), \quad (**)$$

We can assume that $g(x) = 0$ on E_n. We shall say that u_t is of the class S if for each $t \in (0,T], 0 < T \in E_1, u_t \in W^{2m}(E_n)$ and $\frac{du_t}{dt} \in L_2(E_n)$, where $\frac{du}{dt}$ is the abstract derivative of u_t in $L_2(E_n)$, in other word there is and element $v_t \in L_2(E_n)$ such that

$$\lim_{h \to 0} \left\| \frac{1}{h}\left(u_{t+h} - u_t\right) - v_t \right\| = 0,$$

Where $\|\|$ is the norm in $L_2(E_n), t \in (0,T]$.

Notice that if $4m > n, u_t \in W^{2m}(E_n), \frac{du_t}{dt} \in W^{2m}(E_n)$, for each $t \in (0,T]$, then the partial derivative $\frac{\partial u_t}{\partial t}$ exists in the usual sense $E_n \times (0,T]$ and $\frac{du_t}{dt} = \frac{\partial u_t}{\partial t}$. in fact according to the embedding theorem [3], [8], we have

$$\max_{Q} \left| \frac{du_t}{dt} - \frac{\Delta u_t}{\Delta t} \right| \le C \left\| \frac{du_t}{dt} - \frac{\Delta u_t}{\Delta t} \right\| W^{2m}(E_n)$$

Where $\Delta u_t = u_t + \Delta t - u_t, C$ is a positive constant,

$$\|g\|^2 W^{2m}(E_n) = \sum_{k=0}^{2m} \sum_{|\alpha|=k} \int_{E_n} \left| D^\alpha g(x) \right|^2 dx,$$

$dx = dx_1 .. dx_n$ and Q is the volume enclosed by the sphere $x_1^2 + ... + x_n^2 = b^2$ letting $\Delta t \to 0$, we get $\frac{\partial u_t}{\partial t} = \frac{du_t}{dt}$ on $Q \times (0,T]$.

Since b is arbitrary, it follows that

$$\frac{\partial u_t}{\partial t} = \frac{du_t}{dt} \text{ on } E_n \times (0,T]$$

Let us suppose that the following assumptions are satisfied;

(a) The coefficients $a_\alpha, |\alpha| = 2m$ are real functions of t, defined on [o,T] and having continuous derivatives $\frac{da_\alpha(t)}{dt}$ on $[0,T]$

(b) The deferential operator $\sum_{|\alpha|=2m} a_\alpha(x,t) D^\alpha$ is uniformly elliptic on [0,T].

(c) The kernels $\{K_\alpha(t,\theta); |\alpha| \le 2m, t, \beta \in [0,T]\}$ are linear bounded operators acting on into it self. It is assumed that these operators are continuous in $(t,\theta), t, \theta \in [0,T]$. Furthermore it is assumed that the (abstract) partial derivatives $\frac{\partial k_\alpha}{\partial \theta}$ exist for all $|\alpha| = 2m$ and represents linear bounded operators on $L_2(E_n)$ which are continuous in $(t,\theta), t, \theta \in [0,T]$.

(d) The coefficients $a_\alpha, |\alpha| < 2m$ are real functions, which are continuous and bounded on $E_n \times [0,T]$.

(e) f_t is a map from [0,T] into $L_2(E_n)$, which is continuous in t with respect to the norm in $L_2(E_n)$.

(f) All the coefficients a_α, for $|\alpha| < 2m$, have continuous bounded partial derivatives $D_r a_\alpha$, on $E_n \times [0,T]; r = 1,..., n$.

(g) $f_t \in W^{2m}(E_n)$

(h) The range of the operators $K_\alpha(t,\theta)(|\alpha| \le 2m, t, \theta \in [0,T])$ is the space $W^{2m}(E_n)$. we assume that all the operators $D_r k_\alpha(t,\theta)$ are bounded , and that $D_r \frac{\partial}{\partial \theta} K_\alpha(t,\theta)$ exist for all $|\alpha| = 2m$.The operators $D_r \frac{\partial}{\partial \theta} K_\alpha(t,\theta)$ are supposed to be bounded and continuous in (t,θ) for all $|\alpha| = 2m$. It is supposed also that $D_r k_\alpha, (t,\theta)$ are continuous in (t,θ)

Proposition 1.

Under conditions (a) , …. , (e) if there is at least one solution in the class S of the Cauchy problem (*), (**), then this solution is the unique such solution.

Proof. If $g \in W^{2m}(E_n), |\alpha| = 2m$, then we have the following representation:

$$D^\alpha g = (-1)^m R^\alpha \nabla^{2m} g, \qquad ...(1)$$

$L_2(E_n)$
Where $R^\alpha = R^{\alpha_1}_1 ... R^{\alpha_n}_n, R_j$ is the singular integral operator defined to be the

$$\lim_{h \to 0} \left\| R_j(h) - R_j) g \right\| = 0,$$

$$R_j(h) g = -i\pi^{-(n+1)/2} \Gamma\left(\frac{n+1}{2}\right) \int_{|x-y|>h} \frac{x_j - y_j}{|x-y|^{n+1}} dy,$$

$$|x|^2 = x_1^2 + ... + x_n^2, i = \sqrt{-1}, \nabla^2 = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$$

and

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt.$$

Notice that $R_j, j = 1,...n$ are bounded operators from $L_2(E_n)$ into itself, [1],[2].

Let $H_1(t)$ be an operator defined by

$$H_1(t) = \sum_{|\alpha|=2m} a_\alpha(t) R^\alpha.$$

According to assumption, the operator $H_1(t)$ has a bounded inverse $H_1^{-1}(t)$ defined on for every $\in_t \in [0,T]$.

Set $H_2(t,\theta) = \sum_{|\alpha|=2m} K_\alpha(t,\theta)R^\alpha$. and

$$H_3(t,\theta) = H_2(t,\theta)H_1^{-1}(\theta).$$

Using (1), then equation (*) can be written in the form;

$$\frac{du_t}{dt} - \sum_{|\alpha|=2m} a_\alpha(t)D^\alpha u = \int_0^t H_3(t,\theta) \sum_{|\alpha|=2m} a_\alpha(t)D^\alpha u_\theta d\theta$$

$$+\int_0^t \sum_{|\alpha|<2m} K_\alpha(t,\theta)D^\alpha u_\theta d\theta + \sum_{|\alpha|<m} a_\alpha(x,t)D^\alpha u_t + f_t \quad ...(2)$$

To Prove the uniqueness of the considered Cauchy problem, we set $f_t = 0$ for all $(x,t) \in E_n \times [0,T]$. Now set

$$\frac{du_t}{dt} - \sum_{|\alpha|=2m} a_\alpha(t)D^\alpha u_t = V_t,$$

Then according to assumptions (1) and (2), we can write

$$u_t(x) = \int_0^t \int_{E_n} G(x-y,t,\theta)V_\theta(y)dyd\theta, \quad ...(3)$$

Where G is the fundamental solution of the Cauchy problem for the parabolic equation

$$\frac{du_t}{dt} = \sum_{|\alpha|=2m} a_\alpha D^\alpha u_t$$

Let $\{U(t,\theta); t,\theta \in [0,T]\}$ be the family of bounded operators defined by

$$U(t,\theta)V_\theta = \int_{E_n} G(x-y,t,\theta)V_\theta(y)dy.$$

Consequently (3) can be written in the form

$$U_t = \int_0^t U(t,\theta)V_\theta d\theta. \quad ...(4)$$

According to the well-known properties of the fundamental solution G, [4], [5], we can see that

$$\|U(t,\theta)g\| \le c\,\|g\|, g \in L_2(E_n), \quad ...(5)$$

$$\|D^\alpha U(t,\theta)g\| \le \frac{C}{(t-\theta)^\gamma}\|g\|, \quad ...(6)$$

For $|\alpha| < 2m, t > \theta, g \in L_2(E_n)$, where C is a positive constant and is a constant satisfying $0 < \gamma < 1$. Substituting from (4) into (2), we get

$$V_t = -\int_0^t H_3(t,\theta)V_\theta d\theta - \int_0^t\int_0^\theta \frac{\partial H_3(t,\theta)}{\partial\theta} U(\theta,\eta)V_\eta d\eta d\theta$$

$$H_3(t,t)\int_0^t U(t,\theta)V_\theta d\theta + \sum_{|\alpha|<2m} \int_0^t\int_0^\theta k_\alpha(t,\theta)D^\alpha U(\theta,\eta)V_\eta d\eta d\theta$$

$$+\sum_{|\alpha|<2m} a_\alpha(x,t)\int_0^t D^\alpha U(t,\theta)V_\theta d\theta \quad ...(7)$$

Using (5) and (6), we get from (7), the following estimation;

$$\|V_t\| \le C\int_0^t \|V_0\| d\theta, \quad ...(8)$$

(To obtain (7) and (8), we already used conditions (c) and (d) where C is a positive constant. Thus (4) and (8) lead immediately to the fact that $u_t(x) = 0$ on $E \times [0,T]$.

Proposition2

Under the condition (a) , …, (h) the solution of the Cauchy problem (*) , (**) exists in the class S.

Proof

Using the conditions from (a) to (e), we obtain

$$V_t = -\int_0^t H_3(t,\theta)V_\theta d\theta - \int_0^t\int_0^\theta \frac{\partial H_3(t,\theta)}{\partial\theta} U(t,\theta)V_\eta d\eta d\theta$$

$$H_3(t,t)\int_0^t U(t,\theta)V_\theta d\theta + \sum_{|\alpha|=2m} \int_0^t\int_0^\theta k_\alpha(t,\theta)D^\alpha U(\theta,\eta)V_\eta d\eta d\theta$$

...(9)

According to (5) and (6), the Volterra integral equation (9) has a unique solution V_t in which satisfies:

where c is a positive constant.

This means that under conditions from (a) to (e), we can obtain the so called mild solution [6] of the Cauchy problem (*) , (**). this solution is represented by

$$u_t = \int_0^t U(t,\theta)V_\theta d\theta$$

Now we must prove that the distributional derivatives $D^\alpha u_t$ exists in $L_2(E_n)$ for all

$$|\alpha| \le 2m, t \in (0,T].$$

To prove that $u_t \in W^{2m}(E_n)\, for$ $t \in (0,T]$, we apply formally the differential operator on both sides of the integral equation (9), then we get;

$$D_\gamma V_t = -\int_0^t D_\gamma H_3(t,\theta)V_\theta d\theta$$

$$-\int_0^t \int_0^\theta D_\gamma \frac{\partial H_3(t,\theta)}{\partial\theta} U(t,\theta)V_\eta d_\eta d\theta$$

$$D_\gamma H_3(t,t)\int_0^t U(t,\theta)V_\theta d\theta$$

$$+\sum_{|\alpha|\le 2m} \int_0^t \int_0^t D_\gamma K_\gamma(t,\theta) D^\alpha U(t,\theta) V_\eta d\eta d\theta$$

$$+\sum_{|\alpha|\le 2m} \alpha(x,t)\int_0^t D^\alpha U(t,\theta) D_\gamma V_\theta d\theta$$

$$+\sum_{|\alpha|\le 2m} (D_r a_\alpha(x,t))\int_0^t D^\alpha U(t,\theta)V_\eta d\theta + D_r f_t \qquad ...(10)$$

Let us consider as the unknown element in the integral equation (10). Under the assumptions (a),…, (h) this integral equation can be solved for $D_r V_t$. Thus $D_r V_t \in L_2(E_n)$, for $t \in (o,T]$. Now we have

$$D_r u_t = \int_0^t D_r U(t,\theta)V_\theta d\theta = \int_0^t U(t,\theta) D_r V_\theta d\theta. \qquad ...(11)$$

Using (11), we can wire

$$D^\alpha u_t = \int_0^t D^\beta U(t,\theta) D_\gamma V_\theta d\theta,$$

Where $|\alpha| = 2m, |\beta| = 2m-1.$

Proposition 3

Let $\{f_t^\gamma\}$ be a sequence of functions which are belonging to $L_2(E_n)$ for every $t \in (0,T]$ and continuous in t. suppose.

$$\lim_{\gamma\to\infty} \sup_t \| f_t^\gamma - f_t \| = 0,$$

Where f_t is continuous in $t \in [0,T]$ if is a sequence of functions of the class S, which are solutions of the Cauchy problem

$$\frac{du_t^\gamma}{dt} = Lu_t^\gamma + f_t^\gamma, u_t^\gamma = 0,$$

then $\{u_t^\gamma\}$ converges in $L_2(E_n)$ to the solution of the Cauchy problem (*), (**).

Proof. Set

$$u_t^\gamma = \int_0^t U(t,\theta)V_\theta^\gamma d\theta$$

We find

$$\| V_t^k - v_t^l \| \le \sup_t \| f_t^k - f_t^l \| e^{ct}.$$

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