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Group analysis and variational principle for nonlinear (3+1) schrodinger equation

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ABSTRACT

The generators of the admitted variational Lie symmetry groups are derived and conservation laws for the conserved currents are obtained via Noether's theorem. Moreover, the consistency of a functional integral are derived for the nonlinear Schrödinger equation. In addition to this analysis functional integral are studied using Lie groups.

Key words: Nonlinear (3+1) schrodinger equation, Noether's theorem, Lie groups.

INTRODUCTION

In recent years, a number of works of the symmetry methods are found to be very efficient in applications to differential equations in Physics and Engineering. A subject of a special interest is a study of invariance properties of the equations with respect to local Lie groups point transformations of dependent and independent variables. The importance of the conservation laws lies in the fact that there are situations where numerical schemes have been devised keeping in view the conservation form of the DEs. Also, the conservation law can be used for serving a priori estimates and to obtain integrals of motion, where for certain types of solutions, the conserved density, when integrated, provides us with a constant of motion of the system. Actually, finding the conservation laws of a system is often the first step towards finding its solution. Rund[1] and Logan[2] have studied the invariance of fundamental functional integral and deduced the first integrals or conservation law for the corresponding system of DEs. The nonlinear (3+1) Schrödinger equation [3, 4] is described by nonlinear couple partial differential equations. It is well know that a part of one parameter symmetry groups of these equations turns out to be their variational symmetries. According to Noether theorem [5, 6] such as invariance of the elementary action is a necessary and sufficient condition of the existence of conservation laws for the Schrödinger equation. Let us consider Schrödinger equation with nonlinear term:

$$i q_t + q_{xx} + q_{yy} + q_{zz} + \left(|q|^2 - S(t,x,y,z)\right) q = 0. \quad ...(1)$$

Group Analysis

Equations (1) have some applications in quantum field theory, plasma physics and Engineering [3]. To simplify equation (1), set $q = u + iv$ then, eq. (1) is divided into couple of equations as follows:

$$N = -v_t + u_{xx} + u_{yy} + u_{zz} + u\ \left(u^2 + v^2 - S(t,x,y,z)\right) \quad ...(2)$$

$$M = -u_t - v_{xx} - v_{yy} - v_{zz} - v\ \left(u^2 + v^2 - S(t,x,y,z)\right). \quad ...(3)$$

In order to find invariance transformations, we look for infinitesimal Lie point transformations of the form:

$$\begin{aligned}\bar{x} &= x + \varepsilon\xi(t,x,y,z,u,v) + O(\varepsilon^2),\\ \bar{y} &= y + \varepsilon\eta(t,x,y,z,u,v) + O(\varepsilon^2),\\ \bar{z} &= z + \varepsilon\alpha(t,x,y,z,u,v) + O(\varepsilon^2),\\ \bar{t} &= t + \varepsilon\tau(t,x,y,z,u,v) + O(\varepsilon^2),\\ \bar{u} &= u + \varepsilon\eta^{(1)}(t,x,y,z,u,v) + O(\varepsilon^2),\\ \bar{v} &= v + \varepsilon\eta^{(2)}(t,x,y,z,u,v) + O(\varepsilon^2).\end{aligned} \quad ...(3)$$

which (2-i, 2-ii) system.

Following the widely used methods in the classical monographs. Concerning these arguments [4-12] we find the coordinates $\xi, \eta, \alpha, \tau, \eta^{(1)}, \eta^{(2)}$ by solving over the determined linear PDE system, usually called the determining system which is obtained by requiring, the invariance of the system (2-i,2-ii) with respect to (3). (1)(2),,,,,......

There are many software packages which aid researchers in obtaining the determining system. But solving it with arbitrary functions requires analysis. We omit the determining system from which we are able to obtain the following results in the form of the coordinates

$$\begin{aligned}\tau &= \lambda_4 t + \lambda_5,\\ \xi &= \frac{1}{2}\lambda_4 x + \lambda_2 y + \lambda_3 z + L_1(t),\\ \eta &= \frac{1}{2}\lambda_4 y - \lambda_2 x + \lambda_1 z + L_2(t),\\ \gamma &= \frac{1}{2}\lambda_4 z - \lambda_1 y - \lambda_3 x + L_3(t),\\ \eta^{(1)} &= -\frac{1}{2}\lambda_4 u + \psi(x,y,z,t)v,\\ \eta^{(2)} &= -\psi(x,y,z,t)u - \frac{1}{2}\lambda_4 v\end{aligned} \quad ...(4)$$

and

$$\psi = -\frac{1}{2}L'_3(t)z - \frac{1}{2}L'_2(t)y - \frac{1}{2}L'_1(t)x + g(t),$$

where, λ_1, λ_2, λ_3 and λ_4 arbitrary constants and $L'_1(t), L'_2(t), L'_3(t)$ are arbitrary functions. Let the function) $S(t, x, y, z,)$ satisfies the equation

$$\frac{1}{2}L''_3(t)z + \frac{1}{2}L''_2(t)y + \frac{1}{2}L''_1(t)x - g'(t) + \lambda_4 S + \tau\frac{\partial S}{\partial t} + \xi\frac{\partial S}{\partial x} + \eta\frac{\partial S}{\partial y} + \gamma\frac{\partial S}{\partial z} = 0$$

Then we have the following cases:

Case 1 When $\lambda_1 = \lambda_2 = \lambda_3$ and $\psi = 0$ then the infinitesimal take the form

$$\tau = \lambda_4 t + \lambda_5, \qquad \xi = \frac{1}{2}\lambda_4 x + L_1,$$

$$\gamma = \frac{1}{2}\lambda_4 z + L_3, \qquad \eta^{(1)} = -\frac{1}{2}\lambda_4 u$$

$$\eta = \frac{1}{2}\lambda_4 y + L_2,$$

$$\eta^{(2)} = -\frac{1}{2}\lambda_4 v$$

and) $S(t, x, y, z,)$ satisfies the equation:

$$\lambda_4 S + \tau\frac{\partial S}{\partial t} + \xi\frac{\partial S}{\partial x} + \eta\frac{\partial S}{\partial y} + \gamma\frac{\partial S}{\partial z} = 0$$

By applying the invariance surface condition, we obtain

$$\theta_1 = \frac{1/4\,\lambda_4 x + L_1}{(\lambda_4 t + \lambda_2)^{1/2}}, \qquad \theta_2 = \frac{1/4\,\lambda_4 y + L_2}{(\lambda_4 t + \lambda_2)^{1/2}},$$

$$\theta_3 = \frac{1/4\,\lambda_4 z + L_2}{(\lambda_4 t + \lambda_2)^{1/2}}$$

$$u = \frac{F(\theta_1,\theta_2,\theta_3)}{(\lambda_4 t+\lambda_5)^{1/2}}, \qquad v = \frac{H(\theta_1,\theta_2,\theta_3)}{(\lambda_4 t+\lambda_5)^{1/2}},$$

$$S = \frac{\mu(\theta_1,\theta_2,\theta_3)}{\lambda_4 t+\lambda_5}.$$

Where λ_1, λ_2, L_1, L_2 are constants and F, H, μ are functions to be determined by substituting (4) into the system (2-i,2-ii) ,that is ,by solving the following reduced system

$$-\frac{1}{2}\lambda_4\left(F+\theta_1 F_{\theta_1}+\theta_2 F_{\theta_2}+\theta_3 F_{\theta_3}\right)+\frac{1}{2}\lambda_4^2\left(H_{\theta_1\theta_1}+H_{\theta_2\theta_2}+H_{\theta_3\theta_3}\right)$$
$$+H\left(F^2+H^2-\mu(\theta_1,\theta_2,\theta_3)\right)=0$$
$$\frac{1}{2}\lambda_4\left(H+\theta_1 H_{\theta_1}+\theta_2 H_{\theta_2}+\theta_3 H_{\theta_3}\right)+\frac{1}{2}\lambda_4^2\left(F_{\theta_1\theta_1}+F_{\theta_2\theta_2}+F_{\theta_3\theta_3}\right)$$
$$+F\left(F^2+H^2-\mu(\theta_1,\theta_2,\theta_3)\right)=0$$

...(6)

In order to solve system (6), we change the variables

$$H = H(\theta), \qquad F = F(\theta), \qquad \theta = \theta_1^2+\theta_2^2+\theta_3^2$$

Consequently, the system (6) takes the form:

$$-\frac{1}{2}\lambda_4\left(F+2\theta F'(\theta)\right)+\frac{1}{2}\lambda_4^2\left(4H'(\theta)+\theta H''(\theta)\right)$$
$$+H\left(F^2+H^2-\mu(\theta)\right)=0,$$

$$\frac{1}{2}\lambda_4\left(H+2\theta H'(\theta)\right)+\frac{1}{2}\lambda_4^2\left(4F'(\theta)+\theta F''(\theta)\right)$$
$$+F\left(F^2+H^2-\mu(\theta)\right)=0,$$

...(7)

from (7), we have

$$F = R\theta^{-1/2}, \qquad H = \frac{-T}{3}\theta^{-3}+N,$$

$$\mu = \frac{R^2}{\theta}-\frac{2T\ N}{3\theta^3}+\frac{T^2}{9\theta^6}+N^2$$

where,,*TRN* are constants.

Case 2

When $\lambda_4=\lambda_3=L_1=L_2=L_3=\psi=0$, then the infinitesimal takes the form

$$\tau=\lambda_5,\ \ \xi=\lambda_2 y, \qquad \eta=-\lambda_2 x$$
$$\gamma=0, \quad \eta^{(1)}=\eta^{(2)}=0. \qquad ...(8)$$

The invariance surface condition may be solved to yield the functional form:

This leads to a reduction of the system (2) in the following form

$$\rho=H(\theta_1,\theta_2,\theta_3), \qquad \varphi=(\theta_1,\theta_2,\theta_3),$$
$$\theta_1=x^2+y^2, \qquad \theta_2=l_1 t-\sin^{-1}\frac{x}{\sqrt{\theta_1}},$$
$$\theta_3=z, \qquad S=\mu(\theta_1,\theta_2,\theta_3)$$

...(10)

and

$$l_1=\frac{\lambda_2}{\lambda_5}$$

This leads to a reduction of the system (2) in the following form

$$l_1 W_{\theta_2}=4H_{\theta_1}+4\theta_1 H_{\theta_1\theta_1}+\frac{1}{\theta_1}H_{\theta_2\theta_2}H_{\theta_3\theta_3}+H\left(H^2+W^2-\mu(\theta_1,\theta_2,\theta_3)\right)$$

$$l_1 H_{\theta_2}=-4W_{\theta_1}-4\theta_1 W_{\theta_1\theta_1}-\frac{1}{\theta_1}W_{\theta_2\theta_2}W_{\theta_3\theta_3}-W\left(H^2+W^2-\mu(\theta_1,\theta_2,\theta_3)\right)$$

We make use of dilation group and after some manipulations on the system (10) we have the equivalent system:

$$4H_{\theta_1}+4\theta_1 H_{\theta_1\theta_1}+H_{\theta_3\theta_3}+H\left(H^2+W^2-\mu(\theta_1,\theta_3)\right)=0,$$
$$4W_{\theta_1}+4\theta_1 W_{\theta_1\theta_1}+W_{\theta_3\theta_3}+W\left(H^2+W^2-\mu(\theta_1,\theta_3)\right)=0,$$

...(10)

Where $H = H(\theta_1,\theta_3),\ W = W(\theta_1,\theta_3),$ using again dilation group on the system (11) one gets

$$F+6\theta^2 F'+4(\theta^3+\theta^2)F''+F\left(F^2+W^{*2}-\mu^*(\theta)\right)=0,$$
$$W^*+6\theta^2 W'^*+4(\theta^3+\theta^2)W''^*+W^*\left(F^2+W^{*2}-\mu^*(\theta)\right)=0,$$

where

$$H = \theta_1^{-1/2} F(\theta), \qquad W = \theta_1^{-1/2} W^*(\theta)$$
$$\mu = \theta_1^{-1/2} \mu^*(\theta), \qquad \theta = \theta_1 \theta_3^{-2}$$

Consequently, system (12) has the solution

$$F = \frac{AB}{\sqrt{\theta}}, \qquad W^* = \frac{A}{\sqrt{\theta}},$$

when $\mu^* = 4 + \frac{B^2}{\theta}\left[A^2 + 1\right]$, A, B, *AB* are constants.

Case 3

In this case the infinitesimal takes the form

$$\tau = \lambda_5, \qquad \xi = L_1(t), \qquad \eta = L_2(t),$$
$$\gamma = L_3(t), \qquad \eta^{(1)} = L_4(t,x,y,z)v, \qquad \eta^{(2)} = -L_4(t,x,y,z)u$$

Applying the rules as before (such as case 1, 2), we get:

$$\theta_1 = \lambda_5 x - \phi_1(t), \qquad \theta_2 = \lambda_5 y - \phi_2(t),$$
$$\theta_3 = \lambda_5 x - \phi_3(t),$$

where

$$\phi_1(t) = \int L_1(t)dt,$$

$$\phi_2(t) = \int L_2(t)dt,$$

$$\phi_3(t) = \int L_3(t)dt,$$

and

$$\frac{\partial L_4(t,x,y,z)}{\partial t} - \lambda_4 \frac{\partial S}{\partial t} - \xi \frac{\partial S}{\partial x} - \eta \frac{\partial S}{\partial y} - \gamma \frac{\partial S}{\partial z} = 0,$$

$$u = H(\theta_1, \theta_2, \theta_3)\ Sin\Phi - F(\theta_1, \theta_2, \theta_3) Cos\Phi,$$
$$v = H(\theta_1, \theta_2, \theta_3)\ Cos\Phi + F(\theta_1, \theta_2, \theta_3) Sin\Phi.$$
$$S = \mu(\theta_1, \theta_2, \theta_3) - \frac{1}{2\lambda_5}\left[\frac{1}{2} z L'_3(t) + \frac{1}{2} y L'_2(t) + \frac{1}{2} x L'_1(t) - 2g(t)\right]$$
$$\Phi(t,x,y,z) = \frac{1}{\lambda_5} \int \left[-\frac{1}{2} z L'_3(t) - -\frac{1}{2} y L'_2(t) - \frac{1}{2} x L'_1(t) + g(t)\right] dt$$

...(13)

Hence ,

$$\Phi = -\frac{1}{2\lambda_5}\left[\frac{1}{2} z L_3(t) + \frac{1}{2} y L_2(t) + \frac{1}{2} x L_1(t) + G(t)\right]$$

where,

$$G(t) = -2\int g(t)dt$$

Using the transformation $H = H(\theta)$, $F = F(\theta)$, $\theta = \theta_1 + \theta_2 + \theta_3$ and system (13) in (2), we obtain,

$$H'' + H\ (F^2 + H^2 - \mu(\theta)) - \frac{M^2}{4\lambda_5^2} H = 0,$$
$$F'' + F\ (F^2 + H^2 - \mu(\theta)) - \frac{M^2}{4\lambda_5^2} F = 0 \qquad ...(14)$$

After some calculations on (14), we have the solution in the form:

$$H = c_1\theta + c_2, \quad F = c_3\theta + c_4$$

provided that $\mu = s_1\theta^2 + s_2\theta + s_3$.

where, $s_1, s_2, s_2, c_1, c_2, c_3, c_4, M$ are constants, and

$$s_1 = c_1 + c_3, \quad s_1 = 2c_1c_2 + 2c_3\ c_4,$$

$$s_3 = c_3^2 + c_4^2 - \frac{M^2}{4\lambda_5^2}$$

Case 4 When $L_1(t), L_2(t)$, $L_3(t)$ and $\psi(t,x,y,z)$ are constant, we get

$$\tau = \lambda_4 t + \lambda_5,$$
$$\xi = \frac{1}{2}\lambda_4 x + \lambda_2 y + \lambda_3 z + L_1,$$
$$\eta = \frac{1}{2}\lambda_4 y - \lambda_2 x + \lambda_1 z + L_2,$$
$$\gamma = \frac{1}{2}\lambda_4 z - \lambda_1 y - \lambda_3 x + L_3,$$
$$\eta^{(1)} = -\frac{1}{2}\lambda_4 u + L_4 v, \qquad ...(15)$$
$$\eta^{(2)} = -L_4 u - \frac{1}{2}\lambda_4 v$$

and

$$\lambda_4 S + \tau \frac{\partial S}{\partial t} + \xi \frac{\partial S}{\partial x} + \eta \frac{\partial S}{\partial y} + \gamma \frac{\partial S}{\partial z} = 0$$

From (15), the invariance surface condition leads to:

$$\theta_1 = (\lambda_4 t + \lambda_5)^{-1/2}\{\lambda_4 x + 2(\lambda_2 y + \lambda_3 z + L_1)\},$$
$$\theta_2 = (\lambda_4 t + \lambda_5)^{-1/2}\{\lambda_4 y + 2(\lambda_1 z - \lambda_2 x + L_2)\},$$
$$\theta_3 = (\lambda_4 t + \lambda_5)^{-1/2}\{\lambda_4 z - 2(\lambda_1 y + \lambda_3 x + -L_3)\},$$

and

$$u = (\lambda_4 t + \lambda_5)^{-1/2}\{s_2 F_1(\theta_1, \theta_2, \theta_3) + s_1 F_2(\theta_1, \theta_2, \theta_3)\},$$
$$v = (\lambda_4 t + \lambda_5)^{-1/2}\{s_1 F_1(\theta_1, \theta_2, \theta_3) + s_2 F_2(\theta_1, \theta_2, \theta_3)\},$$
$$S = \frac{\mu(\theta_1, \theta_2, \theta_3)}{\lambda_4 t + \lambda_5}$$

where $s_1 = \dfrac{2L_4}{4L_4^2 + \lambda_4^2}, \quad s_2 = \dfrac{L_4}{4L_4^2 + \lambda_4^2},$

using (16) and changing the variables

$F_1(\theta_1, \theta_2, \theta_3) = F_1(\theta), \quad F_2(\theta_1, \theta_2, \theta_3) = F_2(\theta),$

$\mu(\theta_1, \theta_2, \theta_3) = \mu(\theta), \quad \theta = \theta_1 + \theta_2 + \theta_3.$

then (2) can be written as the follows

$$\frac{\lambda_4}{2}(s_1F_1 + s_2F_2) + \frac{\lambda_4}{2}\theta(s_1F'_1 + s_2F'_2) + d_1(s_2F''_1 + s_1F''_2) + (s_2F_1 + s_1F_2)\left[(s_1^2 + s_2^2)(F_1^2 + F_2^2) + 4s_1s_2F_1F_2 - \mu(\theta)\right] = 0,$$

$$-\frac{\lambda_4}{2}(s_2F_1 + s_1F_2) - \frac{\lambda_4}{2}\theta(s_2F'_1 + s_1F'_2) + d_1(s_1F''_1 + s_2F''_2) + (s_1F_1 + s_2F_2)\left[(s_1^2 + s_2^2)(F_1^2 + F_2^2) + 4s_1s_2F_1F_2 - \mu(\theta)\right] = 0,$$

and

$$d_1 = 3\lambda_4^2 + 8\left(\lambda_4^2 + \lambda_1^2 + \lambda_3^2 + \lambda_1\lambda_3 - \lambda_1\lambda_2 + \lambda_2\lambda_3\right)$$

System (16) has the solution

$$F_1 = F_2 = \theta^2, \quad \text{and } s_1 + s_2 = 0$$

In this case μ(θ) is an arbitrary function.

variational principle

In order to study variational principle for our problem,

$$N(u,v) = 0$$
$$M(u,v) = 0. \qquad ...(18)$$

The system (18) satisfied the consistency conditions for the existence of functional integral. Consequently, a functional integral can be written by using the formula given by Tonti [8, 9] as:),(vuJ Since

$$\delta J(u,v) = \int_\Omega M(\lambda u, \lambda v)\delta v \, d\Omega + \int_\Omega N(\lambda u, \lambda v)\delta u \, d\Omega$$

or

$$J = \frac{1}{2}\int_\Omega \left(-uv_t - u_x^2 - u_y^2 - u_z^2 + \frac{1}{2}u^2(u^2 + v^2) - u^2 S\right)d\Omega + \frac{1}{2}\int_\Omega \left(vu_t - v_x^2 - v_y^2 - v_z^2 + \frac{1}{2}v^2(u^2 + v^2) - v^2 S\right)d\Omega.$$

since

$$J = \int_\Omega L \, d\Omega \qquad ...(19)$$

with *L* being the Lagrangian function, we obtain

$$L = \frac{1}{2}\left[(vu_t - uv_t) - u_x^2 - u_y^2 - u_z^2 - v_x^2 - v_y^2 - v_z^2 + \frac{1}{2}(u^2 + v^2)^2 - (u^2 + v^2)S(t,x,y,z)\right]$$

for which the Euler-Lagrange equation is

$$\frac{\partial L}{\partial u^r} - D_i \frac{\partial L}{\partial u_i^r} + D_i D_j \frac{\partial L}{\partial u_{ij}^r} + ... = 0.$$

Consequently we have

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial L}{\partial u_z}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right) = 0$$

Hence,

$$-v_t + u(u^2 + v^2) - uS(t,x,y,z) + \frac{\partial}{\partial x}(u_x) + \frac{\partial}{\partial y}(u_y) + \frac{\partial}{\partial z}(u_z) = 0$$

This leads to

$$v_t = u_{xx} + u_{xx} + u_{xx} + u\left((u^2 + v^2) - S(t,x,y,z)\right).$$

Moreover as,

$$\frac{\partial L}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial v_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial v_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial L}{\partial v_z}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial v_t}\right) = 0$$

we obtain

$$u_t + v_{xx} + v_{xx} + v_{xx} + v\left((u^2+v^2) - S(t,x,y,z)\right) = 0$$

In order to prove the invariance of the fundamental functional integral $\int_{\Omega} L\, d\Omega$ where L is the Lagrangian function and Ω represents the domain of integration to be invariant under the one-parameter group of transformations (3), we can use the following theorem:

Theorem 1 [2]

If the fundamental functional integral defined by (19) is invariant under the *f*{*r*parameter family of transformation (3), then the Lagrangian *L* and its derivatives satisfy the *f*{*r*identities:

$$\frac{\partial L}{\partial x^{\alpha}}\tau_s^{\alpha} + \frac{\partial L}{\partial u^{k}}\eta_s^{k} + \frac{\partial L}{\partial \dot{u}_{\alpha}^{k}}\left(\frac{d\eta_s^{k}}{dx^{\alpha}} - \dot{u}_{\nu}^{k}\frac{d\tau_s^{\nu}}{dx^{\alpha}}\right) + \frac{\partial L}{\partial \ddot{u}_{\alpha\beta}^{k}}\left(\frac{d^2\eta_s^{k}}{dx^{\alpha}dx^{\beta}} - \ddot{u}_{\beta\nu}^{k}\frac{d\tau_s^{\nu}}{dx^{\alpha}} - \ddot{u}_{\nu\alpha}^{k}\frac{d\tau_s^{\nu}}{dx^{\beta}}\right.$$
$$\left. - \dot{u}_{\nu}^{k}\frac{d^2\tau_s^{\nu}}{dx^{\alpha}dx^{\beta}} + L\frac{d\tau_s^{\alpha}}{dx^{\alpha}}\right) = 0;$$

for

$$s = 1,2,3,\ldots r, \quad \alpha,\beta = 1,2,3,\ldots..m, \qquad k = 1,2,3,\ldots n$$

where

$$\tau_s^{\alpha} \equiv \frac{\partial \varphi^{\alpha}}{\partial \varepsilon^{s}}\Big|_{\varepsilon=0}, \qquad \eta_s^{k} \equiv \frac{\partial \psi^{k}}{\partial \varepsilon^{s}}\Big|_{\varepsilon=0}$$

and

$$\dot{u}_{\alpha}^{k} = \frac{\partial u^{k}}{\partial x^{\alpha}}, \qquad \ddot{u}_{\alpha\beta}^{k} = \frac{\partial^2 u^{k}}{\partial x^{\beta}\partial x^{\alpha}}$$

Then, Lagrangian and its derivatives must satisfy the condition: L

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x}\xi + \frac{\partial L}{\partial y}\eta + \frac{\partial L}{\partial z}\alpha + \frac{\partial L}{\partial u}\eta^{(1)} + \frac{\partial L}{\partial v}\eta^{(2)} + \frac{\partial L}{\partial u_t}\left\{\frac{\partial \eta^{(1)}}{\partial t} + u_t\left(\frac{\partial \eta^{(1)}}{\partial u} - \frac{\partial \tau}{\partial t}\right) + \frac{\partial \eta^{(1)}}{\partial v}v_t\right.$$
$$-u_t^2\frac{\partial \tau}{\partial u} - u_t v_t\frac{\partial \tau}{\partial v} - u_x\frac{\partial \xi}{\partial t} - u_x u_t\frac{\partial \xi}{\partial u} - u_x v_t\frac{\partial \xi}{\partial v} - u_y\frac{\partial \eta}{\partial t} - u_y u_t\frac{\partial \eta}{\partial u} - u_y v_t\frac{\partial \eta}{\partial v} - u_z\frac{\partial \alpha}{\partial t}$$
$$\left. - u_z u_t\frac{\partial \alpha}{\partial u} - u_z v_t\frac{\partial \alpha}{\partial v}\right\} + \frac{\partial L}{\partial u_x}\left\{\frac{\partial \eta^{(1)}}{\partial x} + u_x\left(\frac{\partial \eta^{(1)}}{\partial u} - \frac{\partial \xi}{\partial x}\right) + \frac{\partial \eta^{(1)}}{\partial v}v_x - u_t\frac{\partial \tau}{\partial x} - u_x u_t\frac{\partial \tau}{\partial u}\right.$$
$$\left. - u_x v_t\frac{\partial \tau}{\partial v} - u_x^2\frac{\partial \xi}{\partial v} - u_x v_x\frac{\partial \xi}{\partial v} - u_y\frac{\partial \eta}{\partial x} - u_y u_x\frac{\partial \eta}{\partial u} - u_y v_x\frac{\partial \eta}{\partial v} - u_z\frac{\partial \alpha}{\partial x} - u_z u_x\frac{\partial \alpha}{\partial u} - u_z v_x\frac{\partial \alpha}{\partial v}\right\}$$
$$+ \frac{\partial L}{\partial u_y}\left\{\frac{\partial \eta^{(1)}}{\partial y} + u_y\left(\frac{\partial \eta^{(1)}}{\partial u} - \frac{\partial \eta}{\partial y}\right) + \frac{\partial \eta^{(1)}}{\partial v}v_y - u_t\frac{\partial \tau}{\partial y} - u_x u_t\frac{\partial \tau}{\partial u} - u_t v_y\frac{\partial \tau}{\partial v} - u_x\frac{\partial \xi}{\partial y}\right.$$
$$\left. - u_x u_y\frac{\partial \xi}{\partial u} - u_x v_y\frac{\partial \xi}{\partial v} - u_y^2\frac{\partial \eta}{\partial u} - u_x v_y\frac{\partial \xi}{\partial v} - u_z\frac{\partial \alpha}{\partial y} - u_z u_y\frac{\partial \alpha}{\partial u} - u_z v_y\frac{\partial \alpha}{\partial v}\right\} + \frac{\partial L}{\partial u_z}\left\{\frac{\partial \eta^{(1)}}{\partial z}\right.$$
$$+ u_z\left(\frac{\partial \eta^{(1)}}{\partial u} - \frac{\partial \alpha}{\partial z}\right) + \frac{\partial \eta^{(1)}}{\partial v}v_z - u_t\frac{\partial \tau}{\partial z} - u_z u_t\frac{\partial \tau}{\partial u} - u_t v_z\frac{\partial \tau}{\partial v} - u_x\frac{\partial \xi}{\partial z} - u_x u_z\frac{\partial \xi}{\partial u} - u_x v_z\frac{\partial \xi}{\partial v}$$
$$\left. - u_y\frac{\partial \eta}{\partial z} - u_y u_z\frac{\partial \eta}{\partial u} - u_y v_z\frac{\partial \eta}{\partial v} - u_z^2\frac{\partial \alpha}{\partial u} - u_z v_z\frac{\partial \alpha}{\partial v}\right\} + \frac{\partial L}{\partial v_t}\left\{\frac{\partial \eta^{(2)}}{\partial t} + v_t\left(\frac{\partial \eta^{(2)}}{\partial v} - \frac{\partial \tau}{\partial t}\right) + \frac{\partial \eta^{(2)}}{\partial u}u_t\right.$$

$$-v_t u_t \frac{\partial \tau}{\partial u} - v_t^2 \frac{\partial \tau}{\partial v} - v_x \frac{\partial \xi}{\partial t} - v_x v_t \frac{\partial \xi}{\partial v} - v_x u_t \frac{\partial \xi}{\partial u} - v_y \frac{\partial \eta}{\partial t} - u_t v_y \frac{\partial \eta}{\partial u} - v_y v_t \frac{\partial \eta}{\partial v} - v_z \frac{\partial \alpha}{\partial t}$$

$$\left. - v_t v_z \frac{\partial \alpha}{\partial v} - u_t v_z \frac{\partial \alpha}{\partial u} \right\} + \frac{\partial L}{\partial v_x}\left\{ \frac{\partial \eta^{(2)}}{\partial x} + v_x \left(\frac{\partial \eta^{(2)}}{\partial v} - \frac{\partial \xi}{\partial x} \right) + \frac{\partial \eta^{(2)}}{\partial u} u_x - v_t \frac{\partial \tau}{\partial x} - v_t v_x \frac{\partial \tau}{\partial v} \right.$$

$$\left. - v_t u_x \frac{\partial \tau}{\partial u} - u_x v_x \frac{\partial \xi}{\partial u} - v_x^2 \frac{\partial \xi}{\partial v} - v_y \frac{\partial \eta}{\partial x} - v_y u_x \frac{\partial \eta}{\partial u} - v_y v_x \frac{\partial \eta}{\partial v} - v_z \frac{\partial \alpha}{\partial x} - u_x v_z \frac{\partial \alpha}{\partial u} - v_x v_z \frac{\partial \alpha}{\partial v} \right\}$$

$$+ \frac{\partial L}{\partial v_y}\left\{ \frac{\partial \eta^{(2)}}{\partial y} + v_y \left(\frac{\partial \eta^{(2)}}{\partial v} - \frac{\partial \eta}{\partial y} \right) + \frac{\partial \eta^{(2)}}{\partial u} u_y - v_t \frac{\partial \tau}{\partial y} - v_t u_y \frac{\partial \tau}{\partial u} - v_t v_y \frac{\partial \tau}{\partial v} - v_x \frac{\partial \xi}{\partial y} - v_x u_y \frac{\partial \xi}{\partial u} \right.$$

$$\left. - v_x v_y \frac{\partial \xi}{\partial v} - v_y u_y \frac{\partial \eta}{\partial u} - v_y^2 \frac{\partial \eta}{\partial v} - v_z \frac{\partial \alpha}{\partial y} - u_y v_z \frac{\partial \alpha}{\partial u} - v_y v_z \frac{\partial \alpha}{\partial v} \right\} +$$

$$+ \frac{\partial L}{\partial v_z}\left\{ \frac{\partial \eta^{(2)}}{\partial z} + v_z \left(\frac{\partial \eta^{(2)}}{\partial v} - \frac{\partial \alpha}{\partial y} \right) + \frac{\partial \eta^{(2)}}{\partial u} u_z - v_t \frac{\partial \tau}{\partial z} - v_t u_z \frac{\partial \tau}{\partial u} - v_t v_z \frac{\partial \tau}{\partial v} - v_x \frac{\partial \xi}{\partial z} - v_x u_z \frac{\partial \xi}{\partial u} - v_x v_z \frac{\partial \xi}{\partial v} - v_y \frac{\partial \eta}{\partial z} - v_y u_z \frac{\partial \eta}{\partial u} - v_z^2 \frac{\partial \alpha}{\partial v} \right.$$

$$\left. - v_y v_z \frac{\partial \eta}{\partial v} - u_z v_z \frac{\partial \alpha}{\partial u} \right\} = 0$$

On Substituting for *L* and its derivatives in equation (21), we get a polynomial in $u_x, u_y, u_z, u_t, v_x, v_y, v_z, v_t,$ etc Collecting it in descending order of various powers of $u_x, u_y, u_z, u_t, v_x, v_y, v_z, v_t, \ldots etc$ and equating to zero the different powers of $u_x, u_y, u_z, u_t, v_x, v_y, v_z, v_t, \ldots etc$, we obtain a system of first order PDEs. On solving this system, we get the following expressions for $\xi, \tau, \eta, \alpha, \eta^{(1)}$ and $\eta^{(2)}$:

$$\xi = -c_1 y - c_2 z + L_1(t);$$
$$\tau = c_4;$$
$$\eta = c_1 x + c_3 z + L_2(t);$$
$$\alpha = c_2 x - c_3 y + L_3(t);$$
$$\eta^{(1)} = -\frac{v}{2}\left[L'_1 x + L'_2 y + L'_3 z - 2L_4\right];$$
$$\eta^{(2)} = \frac{u}{2}\left[L'_1 x + L'_2 y + L'_3 z - 2L_4\right]. \qquad \text{...(22)}$$

and

$$\tau \frac{\partial S}{\partial t} + \xi \frac{\partial S}{\partial x} + \eta \frac{\partial S}{\partial y} + \alpha \frac{\partial S}{\partial z} = -\frac{1}{2}\left[L''_1 x + L''_2 y + L''_3 z\right] + \varphi'(t)$$

where c_1, c_2, c_3 and 4*c*are arbitrary constants, but L_1, L_2, L_3, L_4 and φ arbitrary functions of *t*[12-19].

By using the following theorem

Theorem 2

(Noether‘s Identity)[5,10] Under the hypothesis of theorem 1, the following *f*{*r*conservation laws hold true

$$\frac{d}{dx^{\alpha}}\left[L\tau_s^{\alpha} + \left(\frac{\partial L}{\partial u_{\alpha}^{k}} - \frac{d}{dx^{\beta}} \; \frac{\partial L}{\partial \ddot{u}_{\beta\alpha}^{k}}\right)c_s^{k} + \frac{\partial L}{\partial u_{\beta\alpha}^{k}} \; \frac{d c_s^{k}}{d x^{\beta}}\right] = 0$$

where

$$c_s^k = \eta_s^k - u_\nu^k \tau_s^\nu$$

Consequently

$$\frac{d}{dx^{\alpha}}\left[L\tau_s^{\alpha} + \frac{\partial L}{\partial \dot{u}_{\alpha}^{k}} c_s^{k}\right] = 0$$

thus , we have

$$\frac{\partial}{\partial t}\left\{L\tau + \frac{\partial L}{\partial u_t}c^1 + \frac{\partial L}{\partial v_t}c^2\right\} + \frac{\partial}{\partial x}\left\{L\xi + \frac{\partial L}{\partial u_x}c^1 + \frac{\partial L}{\partial v_x}c^2\right\} + \frac{\partial}{\partial y}\left\{L\eta + \frac{\partial L}{\partial u_y}c^1 + \frac{\partial L}{\partial v_y}c^2\right\}$$
$$+ \frac{\partial}{\partial z}\left\{L\alpha + \frac{\partial L}{\partial u_z}c^1 + \frac{\partial L}{\partial v_z}c^2\right\} = 0,$$

where,

$$c^1 = \eta^{(1)} - u_\nu \tau^\nu = \eta^{(1)} - u_t \tau - u_x \xi - u_z \alpha - u_y \eta,$$
$$c^1 = \eta^{(2)} - v_t \tau - v_x \xi - v_z \alpha - v_y \eta.$$

Hence

$$\frac{\partial}{\partial t}\left\{\frac{c_4}{2}\left[-(u_x^2+u_y^2+u_z^2)-(v_x^2+v_y^2+v_z^2)+\frac{1}{2}(u^2+v^2)^2-(u^2+v^2)S\right]-\frac{1}{4}(u^2+V^2)(L'_1x+\right.$$

$$L'_2y+L'_3z+L_4)+\frac{1}{2}(Vu_x-uV_x)(c_1y-c_2z-L_1)+\frac{1}{2}(uV_y-Vu_y)(c_1x+c_3z+L_2)+$$

$$\left.\frac{1}{2}\ (uV_z-Vu_z)(c_2x-c_3y-L_3)\right\}+$$

$$\frac{\partial}{\partial x}\left\{\frac{1}{2}(c_1y-c_2z+L_1)\left[(Vu_t-uV_t)+(u_x^2-u_y^2-u_z^2)+(V_x^2-V_y^2-V_z^2)+\frac{1}{2}\left((u^2+V^2)^2-(u^2+V^2)S\right)\right]\right.$$

$$\frac{1}{2}(v\ u_x-uv_x)(L'_1x+L'_1y+L'_1z+L_4)+c_4(u_xu_t+v_xv_t)+(u_xu_y+v_xv_y)(c_2x+c_3z+L_2)+(u_xu_z+v_xv_z)(c_2x-c_3y+L_3)$$

$$+\frac{\partial}{\partial y}\left\{\frac{1}{2}(c_1x+c_3z+L_2)\left[(Vu_t-uV_t)+(u_y^2-u_x^2-u_z^2)+(V_y^2-V_x^2-V_z^2)+\frac{1}{2}(u^2+V^2)^2-(u^2+V^2)S\right]+\right.$$

$$+\frac{1}{2}\ (Vu_y-uV_y)(L'_1\ x+L'_2y+L'_3z-L_4)\ +c_4\ (u_yu_t+V_yV_t)-(u_xu_y+V_xV_y)(c_1y+c_2z-L_1)+$$

$$+(u_yu_z+v_zv_y)(c_2x-c_3y+L_3)+\frac{\partial}{\partial z}\left\{\frac{1}{2}(c_2x-c_3y+L_3)\left[(vu_t-uv_t)+(u_z^2-u_x^2-u_y^2)+(v_z^2-v_x^2-v_y^2)\right.\right.$$

$$\left.+\frac{1}{2}(u^2+v^2)^2-(u^2+v^2)S\quad\right]+\frac{1}{2}\ (vu_z-uv_z)(L'_1\ x+L'_2y+L'_3z-L_4)+c_4\ (u_zu_t+v_zv_t)-$$

$$\left.-(u_xu_z+v_xv_z)(c_1y+c_2z-L_1)+(u_yu_z+v_zv_y)(c_2x+c_3z+L_2)-\right\}=0$$

CONCLUSION

This paper is concerned with finding potentials that admit Lie point symmetries for the nonlinear (3+1) Schrödinger equation, i.e., we give a classification of potentials admitting point symmetries. Finally, we apply Noether's theorem to show which of this point symmetries are variational and we obtain the corresponding conservation laws

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