Symmetric duality for Bonvex multiobjective fractional continuous time programming problems

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ABSTRACT

We introduced a symmetric dual for multiobjective fractional variational programs in second order. Under invexity assumptions, we established weak, strong and converse duality as well as self duality relations. We work with properly efficient solutions in strong and converse duality theorems. The weak duality theorems involves efficient solutions.

Key words: Bonvex multiobjective fractional variational program.

INTRODUCTION

The growing science of communication between two communicators requires security and integrability. After the wide study of fractal sets, Jensen’s inequality for convex functions and it’s usefulness in the study of entropy, paves the way for multiobjective continuous time programming.

The classical dual in linear programming is symmetric in the sense that the dual of dual is the original linear programming. Such symmetry is not found in duality concepts for nonlinear programming, not even in quadratic programming in36. Dorn introduced a different dual for quadratic programming, which is symmetric. Extending these results to general convex programming, Dantzig, Eisenberg and Cottle formulated a symmetric dual and established weak and strong duality relations. Symmetric duality results under generalized convexity were given by Mond and Weir in9 for new types of a dual, then in10 Weir and Mond introduced two distinct symmetric duals for mathematical programming under additional assumptions mathematical programming are shown to be self dual.

In9 Mond and Hanson first extended the symmetric duality results of17 to variational problems by introducing continuous analogues of the earlier concepts. Since Hanson defined invexity in24 as a new generalization of convexity. Several authors have introduced concepts of invexity and generalized invexity for use in convex programming e.g. [1,15,7,22]. Smart and Mond22 extended symmetric duality results to variational programming by employing a continuous version of invexity. Kim and Lee15 presented a symmetric duality in the sense of a dual proposed by Mond and Weir in9 not in the sense pf Wolfe’s dual as in22, establishing duality relations for variational programming here by assuming pseudoinvexity. subsequently, Kim, Lee and Lee extended the results in15 to the mathematical case. More, recently Kim and Lee15 formulated a symmetric and a generalized symmetric dual for mathematical variational programming, weak, strong and converse duality relations are obtained under invexity assumptions. Further generalizations of convexity for continuous time programming have been done by many authors like Kim and Lee15, Mond and Hussain7, Mukherjee and Mishra30. Assumptions of convexity / concavity for functions involved are common in these works.
In this article, we focus on symmetric duality for fractional programming for convex function.

In this article, we introduced a symmetric dual for multiobjective fractional variational programs in second order. Under invexity assumptions, we established weak, strong and converse duality as well as self duality relations. We work with properly efficient solutions in strong and converse duality theorems. The weak duality theorems involves efficient solutions. The article is organized as follows. In section 2 we introduced the multiobjective fractional variational programming and its proposed symmetric duality as well as convexity for such problems. Section 3 contains the weak, strong and converse duality theorems as well as a self duality theorems. Finally in section 4 we specialize these results to the static case and particular cases as discussion and conclusion.

MATERIALS AND METHODS

To compare vectors along the lines, we will distinguish between $\mathbb{R}$ and $\mathbb{R}$ between $\mathbb{R}$ and $\mathbb{R}$ specially, all vectors are in $\mathbb{R}^n$.

Let $I$ be a real interval and $g : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$.

Consider the vector valued function $f(t, x, \dot{x}, y, \dot{y})$, where $x$ and $y$, and $\dot{x}$ and $\dot{y}$ denote the derivatives of $x$ and $y$, respectively, with respect to $t$. Assume that $f$ has continuous fourth-order partial derivatives with respect to $x$ and $y$, and $\dot{x}$ and $\dot{y}$ denote the K x n matrices of first order partial derivatives with respect to $x$ and $y$. While $f_{i\dot{x}}$ and $f_{i\dot{y}}$ denote the K x n matrices of second order partial derivatives.

$$i = 1, 2, \ldots, k.$$ Similarly, $f_{i\dot{x}}$ and $f_{i\dot{y}}$ denote the k x m matrices of first order and second order partial derivatives respectively with respect to $x$ and $y$.

Consider the following multiobjective fractional variational programs.

(MFVP)

$$\min \int_a^b f(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t))dt$$

A point $(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) \in X$ is said to be an efficient (Pareto optimal) solution of (MVP) if for all $(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) \in X$, $x < y$ implies $x^* < y$.

A point $(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) \in X$ is said to be a properly efficient solution of (MVP) if it is sufficient for (MVP) and if there exists a scalar $M > 0$ such that, for all $i \in \{1, 2, \ldots, k\}$
\begin{align*}
&\int_a^b f(t,x(t),y(t),\mathbf{w}(t)) \, dt \\
&\leq M\left(\int_a^b f_j(t,x(t),\mathbf{w}(t),y(t),\mathbf{w}(t)) \, dt\right) \\
&\leq M\left(\int_a^b f_j(t,x^*(t),\mathbf{w}(t),y^*(t),\mathbf{w}(t)) \, dt\right) \\
&\text{for some } j, \text{ such that} \\
&\int_a^b f_j(t,x^*(t),\mathbf{w}(t),y^*(t),\mathbf{w}(t)) \, dt > 0 \\
\text{whenever } (t,x(t),\mathbf{w}(t),y(t),\mathbf{w}(t)) \in X 
\end{align*}

Definition

**Bonvexity**

The vector functional \(\int_a^b f\) is Bonvex in \(x\) and, if for each \(y : [a,b] \rightarrow \mathbb{R}^n\) with piecewise smooth on \([a,b]\), then there exist \(A, P : [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n\) such that for all \(i = 1, 2, \ldots, n\).

\[
\int_a^b f(t,x(t),y(t),\mathbf{w}(t)) \, dt \\
\leq \int_a^b f_j(t,x(t),y(t),\mathbf{w}(t)) \, dt \\
\leq \int_a^b f_j(t,x^*(t),y^*(t),\mathbf{w}(t)) \, dt \\
\text{for all } x : [a,b] \rightarrow \mathbb{R}^n, u : [a,b] \rightarrow \mathbb{R}^n \text{ with are piecewise smooth} [a,b].
\]

Definition

**Boncavity**

The vector functional \(\int_a^b f\) functional is Boncave in \(y\) and, if for each \(x : [a,b] \rightarrow \mathbb{R}^m\) with piecewise smooth on \([a,b]\), then there exist \(\xi, W, S : [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n\) such that for all \(i = 1, 2, \ldots, n\).

\[
\int_a^b f(t,x(t),y(t),\mathbf{w}(t)) \, dt \\
\leq \int_a^b f(t,x(t),y(t),\mathbf{w}(t)) \, dt \\
\leq \int_a^b f(t,x(t),y(t),\mathbf{w}(t)) \, dt \\
\text{for all } v : [a,b] \rightarrow \mathbb{R}^m, y : [a,b] \rightarrow \mathbb{R}^m \text{ with are piecewise smooth} [a,b].
\]

Definition

**Pseudo-Bonvexity**

The vector functional is Pseudo-Bonvex in \(x\) and, if for each \(y : [a,b] \rightarrow \mathbb{R}^m\) with piecewise smooth on \([a,b]\), then there exist \(A, P : [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^n\) such that for all \(i = 1, 2, \ldots, n\).

\[
\int_a^b f(t,x(t),y(t),\mathbf{w}(t)) \, dt \\
\leq \int_a^b f_j(t,x(t),y(t),\mathbf{w}(t)) \, dt \\
\leq \int_a^b f_j(t,x(t),y(t),\mathbf{w}(t)) \, dt \\
\text{for all } x : [a,b] \rightarrow \mathbb{R}^n, u : [a,b] \rightarrow \mathbb{R}^n \text{ with are piecewise smooth} [a,b].
\]
Definition 2.5 Pseudo-Boncavity

The vector functional is Pseudo-Boncave in y and , if for each \( x : [a,b] \rightarrow \mathbb{R}^m \) with piecewise smooth on \([a,b]\), then there exist \( \xi_i \), \( W,S : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) such that for all \( i = 1,2,\ldots,n \):

\[
\frac{\partial}{\partial y} \sum_{i=1}^{n} 2(y - \xi_i) (W_i^T f_y(t,x(t),\xi_i) + S_i^T f_x(t,x(t),\xi_i)) \leq 0
\]

for all \( v : [a,b] \rightarrow \mathbb{R}^m \), \( y : [a,b] \rightarrow \mathbb{R}^m \) with \( v(t) \) and \( y(t) \) are piecewise smooth \([a,b]\).

Definition 2.6 (Skew-Symmetric)

The vector functional \( b \int a f \) is said to be skew-Symmetric when both \( x \) and \( y \in \mathbb{R}^n \):

\[
\left( b \int a f \right)(x,y) = \left( b \int a f \right)(y,x)
\]

We make the following assumptions .

(i) \( f \) and \( g \) are thrice continuously differentiable with respect to \((x, (t))\) and \((y, (t))\), \( x : I \rightarrow \mathbb{R}^n \) and \( y : I \rightarrow \mathbb{R}^m \) are piecewise thrice continuously differentiable ;

Let \( (t, x(t), (t), y(t), (t)), \) \( (t, x(t), (t), y(t), (t)), \) etc. ;

(ii) \( F_i = \int_a^b g_i(t,x(t),\xi(t), y(t), \xi(t))dt \) are convex in x and \( \xi \) and –

\[
\frac{\partial}{\partial \xi} (D^2 f_{\xi \xi}) = f_{\xi \xi} + Df_{\xi \xi} \quad \frac{\partial}{\partial y} (D^2 f_y) = Df_{y y}
\]

and \( G_i = \int_a^b g_i(t,x(t),\xi(t), y(t), \xi(t))dt \) are convex in y and ;

(iii) In the problem Multiobjective fractional programming Primal (MFP) and Multiobjective fractional programming Dual (MFD) , the numerators are nonnegative and denominators are positive ; Denote by \( X \) the space of thrice continuously differentiable functions \( x : I \rightarrow \mathbb{R}^n \) with the norm \( \|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2x\|_\infty + \|D^3x\|_\infty \)

where the differential operator \( D \) is given by

\[
u = Dx \Leftrightarrow x(t) = \alpha + \int_a^t u(s)ds, \text{ and } x(a) = \alpha, x(b) = \beta
\]

are given boundary values; thus \( D = \frac{d}{dt} \) except at discontinuity . Denote by \( Y \), the space of thrice continuous differentiable functions \( y : I \rightarrow \mathbb{R}^m \) with the norm similar to that of space \( X \).

All the above statements for \( F \) and \( G \) will be assumed to hold for subsequent results . It is to be noted here that

\[
\frac{\partial}{\partial \xi} (D^2 f_{\xi \xi}) = f_{\xi \xi} + Df_{\xi \xi} \quad \frac{\partial}{\partial y} (D^2 f_y) = Df_{y y}
\]
\[
\frac{\partial}{\partial x} (Df_{(x,y)}) = Df_{(y,x)}, \quad \frac{\partial}{\partial y} (Df_{(x,y)}) = Df_{(y,x)}.
\]

Similarly with \( g \) also.

In order to simplify notations we introduce \((MFP)\)

\[
\begin{align*}
\int_{\alpha}^{\beta} \int_{\gamma}^{\delta} [A f(t,u,v,w), \theta - D^2 f(t,u,v,w), \phi ] dt = f(t,u,v,w,\theta) \quad &\text{subject to} \\
\int_{\alpha}^{\beta} \int_{\gamma}^{\delta} [B g(t,u,v,w), \phi - D^2 g(t,u,v,w), \gamma ] dt = g(t,u,v,w,\phi) \quad &\text{subject to} \\
\end{align*}
\]

\[
x(t) = 0 = x(\beta), \quad y(t) = 0 = y(\beta).
\]

Basic Results

Now, we express \((MFP)\) and \((MFD)\) equivalently as \((MFP^*)\)

\[
\begin{align*}
\min & \quad \int_{\alpha}^{\beta} \int_{\gamma}^{\delta} [A f(t,u,v,w), \theta - D^2 f(t,u,v,w), \phi ] dt = f(t,u,v,w,\theta) \quad &\text{subject to} \\
\int_{\alpha}^{\beta} \int_{\gamma}^{\delta} [B g(t,u,v,w), \phi - D^2 g(t,u,v,w), \gamma ] dt = g(t,u,v,w,\phi) \quad &\text{subject to} \\
\end{align*}
\]

\[
x(t) = 0 = x(\beta), \quad y(t) = 0 = y(\beta).
\]

Max.

\[
\begin{align*}
\int_{\alpha}^{\beta} \int_{\gamma}^{\delta} [A f(t,u,v,w), \theta - D^2 f(t,u,v,w), \phi ] dt = f(t,u,v,w,\theta) \quad &\text{subject to} \\
\int_{\alpha}^{\beta} \int_{\gamma}^{\delta} [B g(t,u,v,w), \phi - D^2 g(t,u,v,w), \gamma ] dt = g(t,u,v,w,\phi) \quad &\text{subject to} \\
\end{align*}
\]

\[
x(t) = 0 = x(\beta), \quad y(t) = 0 = y(\beta).
\]
subject to
\[ u(a) = 0 = u(b), \quad v(a) = 0 = v(b) \] ... (10)

and
\[ \dot{u}(a) = 0 = \dot{u}(b), \quad \dot{v}(a) = 0 = \dot{v}(b) \] ... (11)

\[ -L_i^T = 0 \quad \ldots \text{(13)} \]

\[ \leq 0 \quad \ldots \text{(14)} \]

\[ \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt - \int_a^b \{ f_i(t, u(t), \dot{u}(t), \mathbf{q}_u) - L_i g_i(t, u(t), \mathbf{q}_u) \} dt \geq \]

\[ b \eta(x,u)^T [ \{ f_{ix}(t, u(t), \dot{u}(t), \mathbf{q}_{ux}) - L_{ix} g_{ix}(t, u(t), \mathbf{q}_{ux}) \} + D_i \{ f_{ixx}(t, u(t), \dot{u}(t), \mathbf{q}_{uxx}) - L_{ixx} g_{ixx}(t, u(t), \mathbf{q}_{uxx}) \} dt ] \]

\[ b \eta(x,u)^T [ \{ f_{ix}(t, u(t), \dot{u}(t), \mathbf{q}_{ux}) - L_{ix} g_{ix}(t, u(t), \mathbf{q}_u) \} ] + \]

\[ b \eta(x,u)^T [ \{ f_{ix}(t, u(t), \dot{u}(t), \mathbf{q}_{ux}) - L_{ix}(t, u(t), \mathbf{q}_u) \} ] + \]

\[ b \eta(x,u)^T [ \{ f_{ix}(t, u(t), \dot{u}(t), \mathbf{q}_{ux}) - L_{ix} g_{ix}(t, u(t), \mathbf{q}_u) \} ] + \]

\[ + b \eta(x,u)^T [ \{ f_{ix}(t, u(t), \dot{u}(t), \mathbf{q}_{ux}) - L_{ix} g_{ix}(t, u(t), \mathbf{q}_u) \} ] + \]

\[ + b \eta(x,u)^T [ \{ f_{ix}(t, u(t), \dot{u}(t), \mathbf{q}_{ux}) - L_{ix} g_{ix}(t, u(t), \mathbf{q}_u) \} ] + \]

\[ + b \eta(x,u)^T [ \{ f_{ix}(t, u(t), \dot{u}(t), \mathbf{q}_{ux}) - L_{ix} g_{ix}(t, u(t), \mathbf{q}_u) \} ] + \]

From (8), (14), and (15) with \( \eta+u(t) \geq 0 \), we obtain

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

The invariance assumptions of \( -\int_a^b f_i \) and

\[ \lambda_i > 0, \lambda^T \mathbf{e} = 1, i \in I. \] ... (16)

In the above problem (MFP*) and (MFD*), it is to be noted that I and L are non-negative as a consequence of assumption (iii).

**Duality Theorems**

In this section, we state duality theorems for (MFP*) and (MFD*) which lead to corresponding relations between (MFP) and (MFD). We establish weak, strong and converse duality as well as self-duality relations between (MFP*) and (MFD*).

**Theorem 3.1 (Weak Duality)**

Let \( (x(t), v(t), 1, \lambda) \) be feasible for (MFP*) and let \( (u(t), v(t), L, \lambda) \) be feasible for (MFD*). Assume that \( \int_a^b f_i \) and are invex in \( x \) and

\[ +u(t) \geq 0 \quad \text{and} \quad \mathbf{q}(x, y) + y(t) \geq 0 \quad \text{for all} \ t \in I \quad (\text{except possibly at corners of} \quad (\mathbf{x}(t), \quad (t)) \quad \text{or} \quad (t), \quad (t)). \] Then one has

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

\[ \sum_{i=1}^k \lambda_i \int_a^b \{ f_i(t, x(t), \dot{x}(t), \mathbf{q}_x) - L_i g_i(t, x(t), \mathbf{q}_x) \} dt \geq \]

The invariance assumptions of \( -\int_a^b f_i \) and

\[ \lambda_i > 0, \lambda^T \mathbf{e} = 1, i \in I. \] ... (16)
are Bonvex. Then we have
\[
\int_a^b \{ f_i(t, x, \mathcal{X} v, S) - l g_i(t, x, \mathcal{X} v, S) \} \, dt - \\
\int_a^b \{ f_i(t, x, \mathcal{X} y, S) - l g_i(t, x, \mathcal{X} y, S) \} \, dt \\
\leq \\
\int_a^b \{ f_i(t, x, \mathcal{X} y, S) - l g_i(t, x, \mathcal{X} y, S) \} \, dt \\
- D_i \{ f_i(t, x, \mathcal{X} y, S) - l g_i(t, x, \mathcal{X} y, S) \} \, dt \\
+ \\
\int_a^b \{ f_i(t, x, \mathcal{X} y, S) - D_i g_i(t, x, \mathcal{X} y, S) \} \, dt \\
- l_i \{ g_i(t, x, \mathcal{X} y, S) - D_i g_i(t, x, \mathcal{X} y, S) \} \, dt \\
+ \\
\int_a^b \{ f_i(t, x, \mathcal{X} y, S) - D_i g_i(t, x, \mathcal{X} y, S) \} \, dt \\
- l_i \{ g_i(t, x, \mathcal{X} y, S) - D_i g_i(t, x, \mathcal{X} y, S) \} \, dt \\
\]
Then \((x(t), y(t), R^0, S^0, \lambda_0)\) satisfy the constraints of dual problem and \(l_0 (x(t), y(t), A^0, B^0, \lambda_0) = l_0 (x(t), y(t), R^0, S^0, \lambda_0)\). If in addition, the functional
\[\eta(x) = u(t) \geq 0, \quad \xi(y) = y(t) \geq 0, \quad \{\text{except} \ (x(t), y(t), R^0, S^0, \lambda_0) \}
\]

\[\Rightarrow \quad \lambda^0 \quad \text{(26)}\]

Now, from (20) is
\[\text{Proof}\]

Since \((x(t), y(t), A^0, B^0, l_0)\) is an efficient solution for \((MFP^*)\), therefore, there exist \(\in R^n \times R^m \times R^p\) such that

\[S^1 = \]

Satisfies the following Fritz John conditions

\[S^1_y - DS^1_y + D^2 S^1_y - D^3 S^1_y = 0 \quad \text{... (20)}\]

\[S^1_x - DS^1_x + D^2 S^1_x - D^3 S^1_x = 0 \quad \text{... (21)}\]

\[\{(f_w, e^t - l_0 g, B^t) - D(f_w e^x - l_0 g, B^x)\} (\alpha_i - \beta_i + \mu_i e^t) = 0 \quad \text{... (22)}\]

\[x^0 f_0 \omega_i = 0 \quad \text{... (23)}\]

\[\text{... (24)}\]

\[\sum_{i=1}^{k} \lambda_i (f_w - l_0 g) - D(f_w e^x - l_0 g, B^x) + \mu_i e^t (f_w e^x - l_0 g, B^x) - D(f_w e^x - l_0 g, B^x) = 0 \quad \text{... (25)}\]

\[\{\alpha_i, \beta_i, \mu_i, \omega_i, \lambda_0\} > 0 \quad \text{... (26)}\]

From (21) is

\[\sum_{i=1}^{k} \lambda_i (f_w - l_0 g) - D(f_w e^x - l_0 g, B^x) + \mu_i e^t (f_w e^x - l_0 g, B^x) - D(f_w e^x - l_0 g, B^x) = 0 \quad \text{... (29)}\]

\[\sum_{i=1}^{k} \lambda_i (f_w - l_0 g) - D(f_w e^x - l_0 g, B^x) + \mu_i e^t (f_w e^x - l_0 g, B^x) - D(f_w e^x - l_0 g, B^x) = 0 \quad \text{... (30)}\]

Since \:\{f_w - l_0 g, B^t\} - D^2 (f_w e^x - l_0 g, B^x)\} is non-singular, therefore, (22) yields

\[\alpha_i (A^0 + B^0) - \beta_i + \mu_i e^t = 0 \quad \text{... (31)}\]

We claim that \(\lambda_0 > 0\). \(\text{... (32)}\)

If \(\lambda_0 = 0\) (28) gives
\( i \gamma = \ldots (33) \)

From (33) and (28), we get
\( \sum_{i}^{n} \left( f_{i} - l_{0} \right) g_{i} \geq 0 \),
\( \Rightarrow \)
\( \ldots (34) \)

From hypothesis (ii), equation (34) gives \( \mu_{i} = 0, \beta = 0 \). Hence if \( \alpha = 0 \), then too.

Again from equation (33) and (29), we get \( \omega = 0 \). Hence, we observe that \( \alpha = 0 \), which contradict (27).

Therefore (26) and (25) yield
\( \ldots (35) \)

Now from equation (28) and (31)
\( \ldots (36) \)

Pre-multiplying (36) by \( \alpha(A^{0T} + B^{0T}) \) and using (35), (31) and hypothesis (iii) we get
\( \alpha(A^{0T} + B^{0T}) = 0 \Rightarrow (A^{0T} + B^{0T}) = 0 \)
\( \Rightarrow (A^{0}) = 0, \text{and} \quad (B^{0}) = 0, \text{because} \quad > 0 \).
The proof is analogous to that of theorem 3.2.

A converse duality theorem is stated now.

**Theorem 3.3**

Let \((x_i(t), y_i(t), l_i, \lambda_i)\) be a properly efficient solution for (MFD*). Fix for \(\lambda = \lambda_i\) in (MFP*) and define

\[
\frac{\int g(x, y, t) dt}{\lambda} - \frac{\int f(x, y, t) dt}{\lambda} \quad \text{for} \quad i = 1, 2, ..., k.
\]

Assume that (I)

\[
\int g(x, y, t) dt = \frac{\int f(x, y, t) dt}{\lambda}
\]

implies \(\phi(l(t)) = 0\), for all \(l \in I\), and

(II) \(\int_a^b (f_{i1} - l_{i0} g_{i1}) dt, \int_a^b (f_{i2} - l_{i0} g_{i2}) dt, ..., \int_a^b (f_{ik} - l_{i0} g_{ik}) dt\) is linearly independent. If the invexity conditions of Theorem 3.1 are satisfied, then \((x_i(t), y_i(t), l_i, \lambda_i)\) is properly efficient for (MFP*)

In order to present a better view of the concept of second order self-duality, we will consider here problems (MFP) and (MFD) instead of their equivalents (MFP*) and (MFD*). Assume that \(x(t)\) and \(y(t)\) have the same dimensions, i.e. \(m = n\) the functions will said to be skew symmetric if

\[
f(t, x, y, z) = -f(t, y, x, z)
\]

for all \(x, y, z\) in the domain of \(f\) and the function will be called symmetric if

\[
g(t, x, y, z) = g(t, y, x, z)
\]

in the domain of \(g\).

**Theorem 3.4** (Self-Duality)

If \(f(t, x, y, z)\) is skew symmetric and \(g(t, x, y, z)\) is symmetric, then (MFP) and (MFD) are self-dual. If (MFP) and (MFD) are dual problems, then with \((x_i(t), y_i(t), l_i, \lambda_i)\) also \((y_i(t), x_i(t), l_i, \lambda_i)\) is a joint optimal solution and the common optimal value is 0.
\[
\lambda > 0, \lambda^T e = 1, t \in I.
\]

Which is just the primal problem (MFP).

Thus, if \((x_0(t), y_0(t), l_0, \lambda_0)\) is an optimal solution of (MFD), then \((y_0(t), x_0(t), l_0, \lambda_0)\) is an optimal solution of (MFD).

Since \(f\) is skew-symmetric and \(g\) is symmetric, respectively, we have

\[
l_0(t, y_0, y_0, x_0, x_0) = -l_0(t, x_0, x_0, y_0, y_0)
\]

and so

\[
l_0(t, y_0, y_0, x_0, x_0) = 0.
\]

**REFERENCES**


