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Symmetric duality for Bonvex multiobjective fractional continuous time programming problems

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ABSTRACT

We introduced a symmetric dual for multiobjective fractional variational programs in second order. Under invexity assumptions, we established weak, strong and converse duality as well as self duality relations .We work with properly efficient solutions in strong and converse duality theorems. The weak duality theorems involves efficient solutions .

Key words: Bonvex multiobjective fractional variational program.

INTRODUCTION

The growing science of communication between two communicators requires security and integrability. After the wide study of fractal sets, Jensen's inequality for convex functions and it's usefulness in the study of entropy, paves the way for multiobjective continuous time programming.

The classical dual in linear programming is symmetric in the sense that the dual of dual is the original linear programming . Such symmetry is not found in duality concepts for nonlinear programming , not even in quadratic programming[27]. In[36] Dorn introduced a different dual for quadratic programming , which is symmetric . Extending these results to general convex programming . Dantzig , Eisenberg and Cottle[17] formulated a symmetric dual and established weak and strong duality relations. Symmetric duality results under generalized convexity were given by Mond and Weir in[9] for new types of a dual , then in[33] Weir and Mond introduced two distinct symmetric duals for mathematical programming under additional assumptions mathematical programming are shown to be self dual.

In[8] Mond and Hanson first extended the symmetric duality results of[17] to variational problems by introducing continuous analogues of the earlier concepts . Since Hanson defined invexity in[24] as a new generalization of convexity . Several authors have introduced concepts of invexity and generalized invexity for use in convex programming e.g. [1,15,7,22]. Smart and Mond[22] extended symmetric duality results to variational programming by employing a continuous version of invexity . Kim and Lee[15] presented a symmetric duality in the sense of a dual proposed by Mond and Weir[9] not in the sense pf Wolfe's dual as in[22], establishing duality relations for variational programming here by assuming pseudoinvexity . subsequently , Kim, Lee and Lee extended the results in[15] to the mathematical case .More, recently Kim and Lee[15] formulated a symmetric and a generalized symmetric dual for mathematical variational programming , weak , strong and converse duality relations are obtained under invexity assumptions. Further generalizations of convexity for continuous time programming have been done by many authors like Kim and Lee[15] , Mond and Hussain[7] Mukherjee and Mishra[30]. Assumptions of convexity / concavity for functions involved are common in these works .

In this article , we focus on symmetric duality for fractional programming for bonvex function.

In this article , we introduced a symmetric dual for multiobjective fractional variational programs in second order . Under invexity assumptions , we established weak , strong and converse duality as well as self duality relations .We work with properly efficient solutions in strong and converse duality theorems. The weak duality theorems involves efficient solutions .The article is organized as follows. In section 2 we introduced the multiobjective fractional variational programming and it's proposed symmetric duality as well as bonvexity for such problems. Section 3 contains the weak , strong and converse duality theorems as well as a self duality theorems. Finally in section 4 we specialize these results to the static case and particular cases as discussion and conclusion.

MATERIALS AND METHODS

To compare vectors along the lines, we will distinguish between £ and £ between ³ and ³ specially, all vectors are in R^n.

$x < y \Leftrightarrow x_i < y_i, i = 1, 2, .,.,.n$;

$x \leqq y \Leftrightarrow x_i \leqq y_i, i = 1, 2, .,.,.n$;

$x \leq y \Leftrightarrow x_i \leqq y_i, i = 1, 2, .,.,.n$, but $x \neq y$.

is the negation of .

Let be a real interval and $g : I \times R^n \times R^n \times R^m \times R^m \to R^k$.

Consider the vector valued function $f(t, x, \dot{x}, y, \dot{y})$, where , and are functions of with and , and $\dot{x}$ and denote the derivatives of and , respectively, with respect to t . Assume that f has continuous fourth - order partial derivatives with respect to x and y, and . and denote the K x n matrices of first order partial derivatives with respect to x and . While and s denotes k x n matrices of second order partial derivatives .

and = ,

= , $f_{i\dot{x}\dot{x}}$ =

i =1,2,.,.,k. Similarly, , and ,

denote the k x m matrices of first order and second order partial derivatives respectively with respect to and .

Consider the following multiobjective fractional variational programs.

(MFVP)

min.

$g : I \times R^n \times R^n \times R^m \times R^m \to R^k$

A point $(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t))$ $\in$ X is said to be an efficient (Pareto optimal)solution of (MVP)if for all $(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ $\in$ X ,

>

$$\int_a^b f(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t))dt$$

Definition

A point $(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ $\in$ X is said to be a properly efficient solution of (MVP) if it is sufficient for (MVP) and if there exists a scalar M > 0 such that, for all i$\in${1,2,.,.,k}

–

$$\int_a^b f_i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))dt$$

$-$

$$\leq M(\int_a^b f_j(t, x(t), \dot{x}(t), y(t), \dot{y}(t))dt -$$

$$\int_a^b f_j(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t))dt)$$

for some j, such that

$$\int_a^b f_j(t, x(t), \dot{x}(t), y(t), \dot{y}(t))dt >$$

$$\int_a^b f_j(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t))dt$$

whenever $(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) \in$ X and

$<$

$$\int_a^b f_i(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t))dt$$

Definition

A point $(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) \in$ X is said to be a weakly efficient solution if there exist no other feasible point
for which

$>$

$$\int_a^b f(t, x(t), \dot{x}(t), y(t), \dot{y}(t))dt$$

Definition
Bonvexity

The vector functional $\int_a^b f$ is Bonvex in x and , if for each y : [a,b] $\rightarrow$ R^m with piecewise smooth on [a,b] , then there exist ,A,P : [a,b] x R^n x R^n x R^m x R^m $\rightarrow$ R^n such that for all i = 1,2,.,.,n .

$$\int_a^b \eta(t, x, \dot{x}, u, \dot{u})^T [\{ f_{ix}(t,u,\dot{u}, y, \dot{y}) - Df_{i\dot{x}}(t,u,\dot{u}, y, \dot{y})\} + \{ f_{iuu}(t,u,\dot{u}, y, \dot{y}) - D^2 f_{i\dot{u}\dot{u}}(t,u,\dot{u}, y, \dot{y})\} P(t, x, \dot{x}, u, \dot{u})]dt$$

for all x : [a,b]$\rightarrow$$R^n$, u :[a,b]$\rightarrow$$R^n$ with with are piecewise smooth [a,b].

Definition
Boncavity

The vector $\int_a^b f$ functional is Boncave in y and , if for each x : [a,b] $\rightarrow$ R^m with piecewise smooth on [a,b] , then there exist ξ, W,S: [a,b] x R^n x R^n x R^m x R^m $\rightarrow$ R^n such that for all i = 1,2,.,.,n .

$$\int_a^b f_i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))dt$$

$$-\int_a^b \xi(t, v, \dot{v}, y, \dot{y})^T [\{ f_{iy}(t,u,\dot{u}, y, \dot{y}) - Df_{i\dot{y}}(t,u,\dot{u}, y, \dot{y})\} + \{ f_{iuu}(t,u,\dot{u}, y, \dot{y}) - D^2 f_{i\dot{u}\dot{u}}(t,u,\dot{u}, y, \dot{y})\} S(t, v, \dot{v}, y, \dot{y})]dt$$

for all v : [a,b]$\rightarrow$$R^m$, y :[a,b]$\rightarrow$$R^m$ $\dot{v}(t)$ with
with are piecewise smooth [a,b] .

Definition
Pseudo-Bonvexity

The vector functional is Pseudo-Bonvex in x and , if for each y : [a,b] $\rightarrow$ R^m with piecewise smooth on [a,b] , then there exist ,A,P : [a,b] x R^n x R^n x R^m x R^m $\rightarrow$ R^n such that for all i = 1,2,.,.,n .

$\geq 0 \Rightarrow$ 0 .

$$\int_a^b [\{ f_i(t, x, \dot{x}, y, \dot{y}) - f_i(t,u,\dot{u}, y, \dot{y})\} + \frac{1}{2} A^T \{ f_{ixx}(t,u,\dot{u}, y, \dot{y}) - D^2 f_{i\dot{x}\dot{x}}(t,u,\dot{u}, y, \dot{y})\} A]dt$$

for all x : [a,b]$\rightarrow$$R^n$, u :[a,b]$\rightarrow$$R^n$ with $\dot{x}(t)$ with

are piecewise smooth [a,b] .

Definition 2.5 Pseudo-Boncavity

The vector functional is Pseudo-Boncave in y and , if for each x : [a,b] → R^m with piecewise smooth on [a,b] , then there exist ξ, W,S: [a,b] x R^n x R^n x R^m x R^m ® R^n such that for all i = 1,2,.,.,n .

≤ 0 ⇒

$$\int_a^b [\{ f_i(t,x,\dot{x},v,\dot{v}) - f_i(t,u,\dot{u},y,\dot{y})\} + \frac{1}{2} W^T \{ f_{iyy}(t,u,\dot{u},y,\dot{y}) - D^2 f_{i\dot{y}\dot{y}}(t,u,\dot{u},y,\dot{y})\} W]dt$$

≤ 0

for all v : [a,b]→R^m , y :[a,b]→R^m with $\dot{v}(t)$ with are piecewise smooth [a,b] .

Definition 2.6 (Skew- Symmetric)

The vector functional is is said to be skew- Symmetric when both x and y ∈R^n

= – .

We make the following assumptions .

(i) f and g are thrice continuously differentiable with respect to (x,) and(y,), x : I →R^n and y : I →R^m are piecewise thrice continuously differentiable ;

Let = (t,x(t), (t),y(t), (t)), = (t, x(t), (t),y(t), (t)), etc.;

(ii) F_i =

and – $\int_a^b g_i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))dt$ are bonvex in x and $\dot{x}$, and –

and G_i = $\int_a^b g_i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))dt$ are bonvex in y and ;

(iii) In the problem Multiobjective fractional programming Primal (MFP) and Multiobjective fractional programming Dual (MFD) , the numerators are nonnegative and denominators are positive ; Denote by X the space of thrice continuously differentiable functions x : I →R^n with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2x\|_\infty + \|D^3x\|_\infty$ where the differential operator D is given by

u = Dx ⇔ x(t) = α + $\int_a^t u(s)ds$, and x(a) = , x(b) = β are given boundary values;thus D= $\frac{d}{dt}$ except at discontinuity . Denote by Y, the space of thrice continuous differentiable functions y : I →R^m with the norm similar to that of space X.

All the above statements for F_i and G_i will ... results ... to ... note here that

and

and consequently

, ,

$$\frac{\partial}{\partial \dot{y}}(D^2 f_{\dot{y}\dot{y}}) = f_{\dot{y}\dot{y}} + Df_{\dot{y}\dot{y}}$$ $\frac{\partial}{\partial}$

,

, , $\frac{\partial}{\partial \dot{y}}(D^2 f_{\dot{y}\dot{y}}) = f_{\dot{y}\dot{y}} + Df_{\dot{y}\dot{y}}$,

$$\frac{\partial}{\partial y}(Df_{\dot{y}}) = Df_{\dot{y}y}$$,

$$, \quad \frac{\partial}{\partial \dot{y}}(Df_{\dot{y}}) = Df_{\dot{y}\dot{y}} \quad , \quad \frac{\partial}{\partial x}(Df_{\dot{y}}) = Df_{\dot{y}x} \quad ,$$

$$, \quad \frac{\partial}{\partial \ddot{x}}(Df_{\dot{y}}) = f_{\dot{y}\dot{y}} \quad .$$

Similarly with g also.

In order to simplify notations we introduce
(MFP)

$$\frac{F_i(x,y)}{G_i(x,y)} = \frac{\int_a^b f_i(t,x,\dot{x},y,\dot{y})dt - \int_a^b \frac{1}{2}\left[A^T\{f_{iyy}(t,x,\dot{x},y,\dot{y}) - D^2 f_{i\ddot{y}\ddot{y}}(t,x,\dot{x},y,\dot{y})\}A\right]dt}{\int_a^b g_i(t,x,\dot{x},y,\dot{y})dt + \int_a^b \frac{1}{2}\left[B^T\{g_{iyy}(t,x,\dot{x},y,\dot{y}) - D^2 g_{i\ddot{y}\ddot{y}}(t,x,\dot{x},y,\dot{y})\}B\right]dt} = l$$

min.

$$\frac{\int_a^b f_i(t,x,\dot{x},y,\dot{y})dt - \int_a^b \frac{1}{2}\left[A^T\{f_{iyy}(t,x,\dot{x},y,\dot{y}) - D^2 f_{i\ddot{y}\ddot{y}}(t,x,\dot{x},y,\dot{y})\}A\right]dt}{\int_a^b g_i(t,x,\dot{x},y,\dot{y})dt + \int_a^b \frac{1}{2}\left[B^T\{g_{iyy}(t,x,\dot{x},y,\dot{y}) - D^2 g_{i\ddot{y}\ddot{y}}(t,x,\dot{x},y,\dot{y})\}B\right]dt} = \frac{F_i(x,y)}{G_i(x,y)} = l_i$$

subject to

x(a) = 0 = x(b) , y(a) = 0 = y(b).

$$\dot{x}(a) = 0 = \dot{x}(b), \dot{y}(a) = 0 = \dot{y}(b)$$

$$\int_a^b y(t)^T \sum_{i=1}^k \lambda_i [\{f_{iy}(t,x,\dot{x},y,\dot{y}) - Df_{i\dot{y}}(t,x,\dot{x},y,\dot{y})\}G_i(x,y) - \{g_{iy}(t,x,\dot{x},y,\dot{y}) - Dg_{i\dot{y}}(t,x,\dot{x},y,\dot{y})\}F_i(x,y)]$$
$$+[\{f_{iyy}(t,x,\dot{x},y,\dot{y}) - D^2 f_{i\ddot{y}\ddot{y}}(t,x,\dot{x},y,\dot{y})\}AG_i(x,y) - \{g_{iyy}(t,x,\dot{x},y,\dot{y}) - D^2 g_{i\ddot{y}\ddot{y}}(t,x,\dot{x},y,\dot{y})\}BF_i(x,y)]$$

$\lambda > 0$, $\lambda^T e = 1, t \in I$.

(MFD)

$$\frac{\int_a^b f(t,u(t),\dot{u}(t),v(t),\dot{v}(t))dt - \int_a^b \frac{1}{2}\left[R^T\{f_{ixx}(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - D^2 f_{i\ddot{x}\ddot{x}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))\}R\right]dt}{\int_a^b g(t,u(t),\dot{u}(t),v(t),\dot{v}(t))dt + \int_a^b \frac{1}{2}\left[S^T\{g_{ixx}(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - D^2 g_{i\ddot{x}\ddot{x}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))\}S\right]dt}$$

Max. = =

subject to

u(a) = 0 = u(b) , v(a) = 0 = v(b).

$$\dot{u}(a) = 0 = \dot{u}(b), \dot{v}(a) = 0 = \dot{v}(b) ,$$

≤ 0 ,

$$\int_a^b u(t)^T \sum_{i=1}^k \lambda_i [\{f_{ix}(t,x,\dot{x},y,\dot{y}) - Df_{i\dot{x}}(t,x,\dot{x},y,\dot{y})\}G_i(x,y) - \{g_{ix}(t,x,\dot{x},y,\dot{y}) - Dg_{i\dot{x}}(t,x,\dot{x},y,\dot{y})\}F_i(x,y)]$$
$$+[\{f_{ixx}(t,x,\dot{x},y,\dot{y}) - D^2 f_{i\ddot{x}\ddot{x}}(t,x,\dot{x},y,\dot{y})\}RG_i(x,y) - \{g_{ixx}(t,x,\dot{x},y,\dot{y}) - D^2 g_{i\ddot{x}\ddot{x}}(t,x,\dot{x},y,\dot{y})\}SF_i(x,y)]$$

≤ 0 ,

$\lambda > 0$, $\lambda^T e = 1, t \in I$.

Basic Results

Now, we express (MFP) and (MFD) equivalently as

(MFP*)

Minimize $l = (l_1, l_2, l_3, ., ., ., l_k)^T$ ...(1)

subject to

[illegible] ...(2)

... (3)

...(4)

$- l_i$ = 0 ...(5)

≤ 0 ... (6)

$$\int_a^b y(t)^T \sum_{i=1}^k \lambda_i [\{f_{iy}(t,x,\dot{x},y,\dot{y}) - Df_{i\dot{y}}(t,x,\dot{x},y,\dot{y})\} - l_i\{g_{iy}(t,x,\dot{x},y,\dot{y}) - Dg_{i\dot{y}}(t,x,\dot{x},y,\dot{y})\}]$$
$$+[\{f_{iyy}(t,x,\dot{x},y,\dot{y}) - D^2 f_{i\ddot{y}\ddot{y}}(t,x,\dot{x},y,\dot{y})\}A - l_i\{g_{iyy}(t,x,\dot{x},y,\dot{y}) - D^2 g_{i\ddot{y}\ddot{y}}(t,x,\dot{x},y,\dot{y})\}B]$$

≤ 0 ...(7)

$\lambda > 0$, $\lambda^T e = 1, t \in I$ . ...(8)

(MFD*)

$L = (L_1, L_2, L_3, ., ., ., L_k)^T$ ...(9)

subject to

u(a) = 0 = u(b) , v(a) = 0 = v(b). ...(10)

$$\dot{u}(a) = 0 = \dot{u}(b), \dot{v}(a) = 0 = \dot{v}(b) \quad ...(11)$$

... (12)

-L_i- =0 ...(13)

≤0 , ...(14)

$$\int_a^b u(t)^T \sum_{i=1}^{k} \lambda_i [\{ f_{ix}(t,x,\dot{x},y,\dot{y}) - D f_{i\dot{x}}(t,x,\dot{x},y,\dot{y})\} - L_i\{g_{ix}(t,x,\dot{x},y,\dot{y}) - D g_{i\dot{x}}(t,x,\dot{x},y,\dot{y})\}]$$
$$+[\{ f_{ixx}(t,x,\dot{x},y,\dot{y}) - D^2 f_{i\dot{x}\dot{x}}(t,x,\dot{x},y,\dot{y})\}R - L_i\{g_{ixx}(t,x,\dot{x},y,\dot{y}) - D^2 g_{i\dot{x}\dot{x}}(t,x,\dot{x},y,\dot{y})\}S]$$

≤0, ...(15)

$\lambda > 0$, $\lambda^T e = 1$, $t \in I$ . ...(16)

In the above problem (MFP*) and (MFD*), it is to be noted that l and L are non-negative as a consequence of assumption (iii) .

Duality Theorems

In this section , we state duality theorems for (MFP*) and (MFD*) which lead to corresponding relations between (MFP) and (MFD) . We establish weak, strong and converse duality as well as self-duality +relations between (MFP*)and (MFD*).

Theorem 3.1(Weak Duality)

Let (x(t),y(t),1,λ) be feasible for (MFP*) and let (u(t),v(t),L,λ) be feasible for (MFD*). Assume that $\int_a^b f_i$ and are invex in x and ,

and and are invex in y and , with

+u(t) 0 ≥, and $\xi(v,y)$ + y(t) ≥0 for all t ∈ I (except possibly at corners of ($\dot{x}$(t), (t)) or ((t), (t)). Then one has

$l_i > L_i$.

Proof

The invexity assumptions of and

imply that , i = 1,2,.,.,.,k, are Bonvex . Then we have

$$\int_a^b \{ f_i(t,x,\dot{x},v,\dot{v}) - L_i g_i(t,x,\dot{x},v,\dot{v})\}dt -$$

$$\int_a^b \{ f_i(t,u,\dot{u},v,\dot{v}) - L_i g_i(t,u,\dot{u},v,\dot{v})\}dt$$

≥

$$\int_a^b \eta(x,u)^T [\{ f_{ix}(t,u,\dot{u},v,\dot{v}) - L_i g_{ix}(t,u,\dot{u},v,\dot{v})\} - D\{ f_{i\dot{x}}(t,u,\dot{u},v,\dot{v}) - L_i g_{i\dot{x}}(t,u,\dot{u},v,\dot{v})\}]dt$$

+

$$\int_a^b \eta(x,u)^T [\{ f_{ixx}(t,u,\dot{u},v,\dot{v})R - L_i g_{ixx}(t,u,\dot{u},v,\dot{v})S\} - D^2\{ f_{i\dot{x}\dot{x}}(t,u,\dot{u},v,\dot{v})R - L_i g_{i\dot{x}\dot{x}}(t,u,\dot{u},v,\dot{v})S\}]dt$$

=

$$\int_a^b \eta(x,u)^T [\{ f_{ix}(t,u,\dot{u},v,\dot{v}) - D f_{i\dot{x}}(t,u,\dot{u},v,\dot{v})\} - L_i\{g_{ix}(t,u,\dot{u},v,\dot{v}) - D g_{i\dot{x}}(t,u,\dot{u},v,\dot{v})\}]dt$$

+

$$\int_a^b \eta(x,u)^T [\{ f_{ixx}(t,u,\dot{u},v,\dot{v})R - D^2 f_{i\dot{x}\dot{x}}(t,u,\dot{u},v,\dot{v})\} R - L_i\{g_{ixx}(t,u,\dot{u},v,\dot{v})S - D^2 g_{i\dot{x}\dot{x}}(t,u,\dot{u},v,\dot{v})\}S]dt$$

From (8), (14),and (15) with η+u(t) ≥ 0 , we obtain

$$\sum_{i=1}^{k} \lambda_i \int_a^b \{ f_i(t,x,\dot{x},v,\dot{v}) - L_i g_i(t,x,\dot{x},v,\dot{v})\}dt \geq$$

$$\sum_{i=1}^{k} \lambda_i \int_a^b \Big[\{ f_i(t,u,\dot{u},v,\dot{v}) - \int_a^b \tfrac{1}{2}\Big[R^T\{f_{ixx}(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - D^2 f_{i\dot{x}\dot{x}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))\}R\Big]$$
$$- L_i\{g_i(t,u,\dot{u},v,\dot{v}) - \int_a^b \tfrac{1}{2}\Big[S^T\{g_{ixx}(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - D^2 g_{i\dot{x}\dot{x}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))\}S\Big]\}\Big]dt$$

...(17)

The invexity assumptions of $-\int_a^b f_i$ and

imply that , i = 1,2,.,.,.,k,

are Bonvex . Then we have

$$\int_a^b \{ f_i(t,x,\dot x,v,\dot v) - l_i g_i(t,x,\dot x,v,\dot v)\}dt - \int_a^b \{ f_i(t,x,\dot x,y,\dot y) - l_i g_i(t,x,\dot x,y,\dot y)\}dt$$

$\leq$

$$\int_a^b \xi(v,y)^T [\{ f_{iy}(t,x,\dot x,y,\dot y) - l_i g_{iy}(t,x,\dot x,y,\dot y)\} - D\{ f_{i\dot y}(t,x,\dot x,y,\dot y) - l_i g_{i\dot y}(t,x,\dot x,y,\dot y)\}]dt$$

$+$

$$\int_a^b \xi(v,y)^T [\{ f_{iyy}(t,x,\dot x,y,\dot y)R - l_i g_{iyy}(t,x,\dot x,y,\dot y)S\} - D^2\{ f_{i\ddot y}(t,x,\dot x,y,\dot y)R - l_i g_{i\ddot y}(t,x,\dot x,y,\dot y)S\}]dt$$

$=$

$$\int_a^b \xi(v,y)^T [\{ f_{iy}(t,x,\dot x,y,\dot y) - D f_{i\dot y}(t,x,\dot x,y,\dot y)\} - l_i\{ g_{iy}(t,x,\dot x,y,\dot y)\} - D g_{i\dot y}(t,x,\dot x,y,\dot y)\}]dt$$

$+$

$$\int_a^b \xi(v,y)^T [\{ f_{iyy}(t,x,\dot x,y,\dot y)R - D^2 f_{i\ddot y}(t,x,\dot x,y,\dot y)\}R - l_i\{ g_{iyy}(t,x,\dot x,y,\dot y)S - D^2 g_{i\ddot y}(t,x,\dot x,y,\dot y)\}S]dt$$

From (6), (7),and (16) with y+y(t) $\geq$ 0, we obtain

$$\sum_{i=1}^{k} \lambda_i \int_a^b \{ f_i(t,x,\dot x,v,\dot v) - l_i g_i(t,x,\dot x,v,\dot v)\}dt$$

$$\sum_{i=1}^{k} \lambda_i \int_a^b \begin{matrix}[\{ f_i(t,x,\dot x,y,\dot y) - \int_a^b \frac{1}{2}\left[A^T\{ f_{iyy}(t,x(t),\dot x(t),y(t),\dot y(t)) - D^2 f_{i\ddot y}(t,x(t),\dot x(t),y(t),\dot y(t))\}A\right] \\ -l_i\{ g_i(t,x,\dot x,y,\dot y) - \int_a^b \frac{1}{2}\left[B^T\{ g_{iyy}(t,x(t),\dot x(t),y(t),\dot y(t)) - D^2 g_{i\ddot y}(t,x(t),\dot x(t),y(t),\dot y(t))\}B\right]\}]dt\end{matrix}$$

...(18)

Combining (17) & (18) along with (5) and (13) gives

$$\sum_{i=1}^{k} \lambda_i (l_i - L_i) \int_a^b g_i(t,x,\dot x,v,\dot v)dt \geq 0 \qquad ...(9)$$

If , for some i , $l_i < L_i$ and for all j i , $l_j \leq L_j$, then $\int_a^b g_i(t,x,\dot x,v,\dot v)dt$ > 0 ,

i = 1,2,.,.,.,k, implies that

$$\sum_{i=1}^{k} \lambda_i (l_i - L_i) \int_a^b g_i(t,x,\dot x,v,\dot v)dt < 0$$

which contradicts (19). Hence $l_i > L_i$.

We now present proof of a strong and a converse duality theorems .

Theorem 3.2 (Strong Duality)

Let be thrice differentiable function on $R^n \times R^m$. Suppose that $(x_0(t),y_0(t),A^0,B^0,\lambda_0)$ be a properly efficient solution for (MFP*). Fix $\lambda=\lambda_0$ in (MFD*) and l_{0i} =

$$\frac{\int_a^b f_i(t,x_0,\dot x_0,y_0,\dot y_0)dt - \int_a^b \frac{1}{2}\left[A^T\{ f_{iyy}(t,x_0,\dot x_0,y_0,\dot y_0) - D^2 f_{i\ddot y}(t,x_0,\dot x_0,y_0,\dot y_0)\}A\right]dt}{\int_a^b g_i(t,x_0,\dot x_0,y_0,\dot y_0)dt + \int_a^b \frac{1}{2}\left[B^T\{ g_{iyy}(t,x_0,\dot x_0,y_0,\dot y_0) - D^2 g_{i\ddot y}(t,x_0,\dot x_0,y_0,\dot y_0)\}B\right]dt} = \frac{F_{i0}(x,y)}{G_{i0}(x,y)} = l_{i0}$$

i= 1,2,.,.,.,k. Assume that

(i) $\sum_{i=1}^{k} \lambda_{0i}[\{ f_{iyy} - l_{0i} g_{iyy}\} - D^2\{ f_{i\ddot y} - l_i g_{i\ddot y}\}]$ is nonsingular,

(ii)

$$\sum_{i=1}^{k} \lambda_{0i}[\{(f_{iy} - D f_{i\dot y}) + (f_{iyy} - D^2 f_{i\ddot y})A^0\} - l_{0i} \{(g_{iy} - D g_{i\dot y}) + (g_{iyy} - D^2 g_{i\ddot y})B^0\}] \neq 0$$

and

(iii) { $\int_a^b (f_{1y} - l_{01} g_{1y})dt$, $\int_a^b (f_{2y} - l_{02} g_{2y})dt$,.,., $\int_a^b (f_{ky} - l_{0k} g_{ky})dt$ } is linearly independent.

If The invexity conditions of Theorem 3.1 are satisfied, then $(x_0(t),y_0(t),\lambda_0 ,\lambda_0)$ is properly efficient for (MFD*) .

(iv) The system

$$\begin{aligned}&\sum_{i=1}^{k} \lambda_{0i} \int_a^b \varphi(t)^T([\tfrac{1}{2}\{(f_{iyyy} - D^2 f_{i\dddot y})A^0 - l_{0i}(g_{iyyyy} - D^2 g_{i\dddot y}) - D f_{i\dot y} + D^2 f_{i\ddot y}\}] \\ &- D\tfrac{1}{2}[\{(f_{iyy\dot y} - D^2 f_{i\dddot y} - D f_{i\ddot y})A^0 + (f_{iy\dot y} - D f_{i\ddot y} - f_{i\dot y})\} - \\ &l_{0i}\{(g_{iyy\dot y} - D^2 g_{i\dddot y} - D g_{i\ddot y})B^0 + (g_{iy\dot y} - D g_{i\ddot y} - g_{i\dot y})\}] \\ &+ D^2[\{-\tfrac{1}{2}(f_{i\ddot y} + D f_{i\dddot y})A^0 + f_{i\dot y}\} - l_{0i}\{-\tfrac{1}{2}(g_{i\ddot y} + D g_{i\dddot y})B^0 + g_{i\dot y}\}] - D^3[-\{ f_{i\dddot y}A^0\} - \{-l_{0i}\{ g_{i\dddot y}B^0\}\}]\varphi(t)) = 0\end{aligned}$$

has only solution $\phi(t) = 0$ for all $t \in [a,b]$.

Then $(x_0(t),y_0(t),R^0,S^0,\lambda_0)$ satisfy the constraints of dual problem and l_{i0} $(x_0(t),y_0(t),A^0,B^0,\lambda_0)$ $=L_{i0}$ $(x_0(t),y_0(t),R^0,S^0,\lambda_0)$. If in addition , the functional

, i = 1,2,.,.,.,k, is pseudo Bon-invex

in x and for all feasible solutions of (MFP*) and pseudo Boncave in y and for all feasible solutions of (MFD*) with

$\eta(x,u)$ +u(t) 0 and $\xi(v,y)$ +y(t) 0 , {except perhaps at corners of ($\dot{x}$(t), (t)) or

((t), (t)) }. Then $(x_0(t),y_0(t),A^0,B^0,\lambda_0)$ is global solution for (MFP*) and $(x_0(t),y_0(t),R^0,S^0,\lambda_0)$ is global solution for (MFD*) .

Proof

Since $(x_0(t),y_0(t),A^0,B^0,l_0)$ is an efficient solution for (MFP*) , therefore , there exist

$\in R^n\times R^n\times R^m\times R^m$ such that

$S^1 =$,

Satisfies the following Fritz John conditions

$$S^1_y - DS^1_{\dot{y}} + D^2 S^1_{\ddot{y}} - D^3 S^1_{\dddot{y}} = 0 \quad \text{...(20)}$$

$$S^1_x - DS^1_{\dot{x}} + D^2 S^1_{\ddot{x}} - D^3 S^1_{\dddot{x}} = 0 \quad \text{...(21)}$$

$$\{(f_{iyy}A^{oT} - l_{0i}g_{iyy}B^{0T}) - D(f_{i\dot{y}\dot{y}}A^{oT} - l_{0i}g_{i\dot{y}\dot{y}}B^{0T})\}(\alpha_i - \beta_i + \mu_i y^{0T}) = 0 \quad (22)$$

$$x^{0T}\omega_i = 0 \quad \text{...(23)}$$

...(24)

$$\sum_{i=1}^{k}\lambda_{0i}\mu_i y^{0T}\{(f_{iy} - l_{0i}g_{iy}) - D(f_{i\dot{y}} - l_{0i}g_{i\dot{y}})\} + \mu_i y^{0T}\{(f_{iyy}A^{0T} - l_{0i}g_{iyy}B^{0T}) - D(f_{i\dot{y}\dot{y}}A^{0T} - l_{0i}g_{i\dot{y}\dot{y}}B^{0T}) = 0 \quad \text{...(25)}$$

$$(\alpha_i, \beta_i, \mu_i, \omega_i, \lambda_{0i}) > 0 . \quad \text{...(26)}$$

$$\neq 0 . \quad \text{... (27)}$$

Now, from (20) is

...(28)

From (21) is

$$\sum_{i=1}^{k}\lambda_{0i}(\alpha-\mu)[\{(f_{ix} - l_{0i}g_{ix}) - D(f_{i\dot{x}} - l_{0i}g_{i\dot{x}})\} - (\tfrac{1}{2}\alpha(A^0+B^0) - \beta + \mu.y^0)$$
$$\{(f_{iyyx}A^{0T} - l_{0i}g_{iyyx}B^{0T}) - D^2(f_{i\dot{y}\dot{y}x}A^{0T} - l_{0i}g_{i\dot{y}\dot{y}x}B^{0T})\}$$
$$+(\beta - \mu.y^{0T})\{(f_{iyx}A^{0T} - l_{0i}g_{iyx}B^{0T}) - D(f_{i\dot{y}x} - l_{0i}g_{i\dot{y}x})\}]$$
$$-D[\left(\sum_{i=1}^{k}\lambda_{0i}\right) - (\tfrac{1}{2}\alpha(A^0+B^0) - \beta + \mu.y^0)\{(f_{iyy\dot{x}}A^{0T} - l_{0i}g_{iyy\dot{x}}B^{0T}) - D(f_{i\dot{y}\dot{y}\dot{x}}A^{0T} - l_{0i}g_{i\dot{y}\dot{y}\dot{x}}B^{0T})$$

[illegible]

$$+\sum_{i=1}^{k}\lambda_{0i}(\beta_i - \mu_i.y^{0T})\{(f_{iy\dot{y}}A^{0T} - l_{0i}g_{iy\dot{y}}B^{0T}) - D(f_{i\dot{y}\dot{y}} - l_{0i}g_{i\dot{y}\dot{y}}) - (f_{i\dot{y}}A^{0T} - l_{0i}g_{i\dot{y}}B^{0T})\}]$$
$$+D^2[(\tfrac{1}{2}\alpha(A^0+B^0) - \beta + \mu.y^0)\{D^2(f_{i\dot{y}\ddot{y}}A^{0T} - l_{0i}g_{i\dot{y}\ddot{y}}B^{0T}) + D(f_{i\ddot{y}\ddot{y}}A^{0T} - l_{0i}g_{i\ddot{y}\ddot{y}}B^{0T})\} \quad \text{...(29)}$$
$$\{\sum_{i=1}^{k}\lambda_{0i}(\beta_i - \mu_i y^{0T})\{-(f_{i\ddot{y}} - l_{0i}g_{i\ddot{y}})\}]$$
$$-D^3[(\tfrac{1}{2}\alpha(A^0+B^0) - \beta + \mu y^0)\{(f_{i\dddot{y}}A^{0T} - l_{0i}g_{i\dddot{y}}B^{0T})\}] = 0$$

$$\sum\lambda_{0i}\int_a^b (F_i - l_{0i}G_i)\,dt = 0 \quad \text{...(30)}$$

Since $\{(f_{iyy} - l_{0i}g_{iyy}) - D^2(f_{i\dot{y}\dot{y}} - l_{0i}g_{i\dot{y}\dot{y}})\}$ is non-singular , therefore ,(22) yields

$$\alpha_i(A^0 + B^0) - \beta_i + \mu_i y^0 = 0$$

$\Rightarrow$ ...(31)

We claim that > 0 . ...(32)

If = 0 (28) gives

...(33)

From (33) and (28) , we get

=0

...(34)

From hypothesis (ii) , equation (34) gives $\mu_i = 0, \beta_i = 0$. Hence if = 0 , then too .

Again from equation (33) and (29) , we get . Hence, we observe that = 0 , then and which contradict (27).

Therefore (26) and (25) yield

...(35)

Now from equation (28)and (31)

$$\sum_{i=1}^{k} \lambda_{0i}(\alpha_i - \mu_i)[\{(f_{iy} - l_{0i}g_{iy}) - D(f_{i\dot{y}} - l_{0i}g_{i\dot{y}})\} + \{(f_{iyy}A^{0T} - l_{0i}g_{iyy}B^{0T}) - D(f_{iy\dot{y}}A^{0T} - l_{0i}g_{iy\dot{y}}B^{0T})\}]$$
$$-(\tfrac{1}{2}\alpha_i(A^0 + B^0))\{(f_{iyyy}A^{0T} - l_{0i}g_{iyyy}B^{0T}) - D^2(f_{iy\ddot{y}}A^{0T} - l_{0i}g_{iy\ddot{y}}B^{0T})\}$$
$$-D[-(\tfrac{1}{2}\alpha_i(A^0 + B^0))\{(f_{iyy\dot{y}}A^{0T} - l_{0i}g_{iyy\dot{y}}B^{0T}) - D(f_{iy\ddot{y}}A^{0T} - l_{0i}g_{iy\ddot{y}}B^{0T}) - D^2(f_{iy\dddot{y}}A^{0T} - l_{0i}g_{iy\dddot{y}}B^{0T})\}$$
$$+\sum_{i=1}^{k} \lambda_{0i}\{(f_{iy\dot{y}}A^{0T} - l_{0i}g_{iy\dot{y}}B^{0T}) - D(f_{i\ddot{y}} - l_{0i}g_{i\ddot{y}}) - (f_{i\dot{y}}A^{0T} - l_{0i}g_{i\dot{y}}B^{0T})\}]$$
$$+D^2[(\tfrac{1}{2}\alpha_i(A^0 + B^0))\{D^2(f_{iy\ddot{y}}A^{0T} - l_{0i}g_{iy\ddot{y}}B^{0T}) + D(f_{iy\dddot{y}}A^{0T} - l_{0i}g_{iy\dddot{y}}B^{0T})\}$$
$$\{\sum_{i=1}^{k} \lambda_{0i})\{-(f_{i\ddot{y}} - l_{0i}g_{i\ddot{y}})\}] - D^3[(\tfrac{1}{2}\alpha(A^0 + B^0))\{(f_{iy\dddot{y}}A^{0T} - l_{0i}g_{iy\dddot{y}}B^{0T})\}] = 0$$

...(36)

Pre-multiplying (36) by $\alpha_i(A^{0T} + B^{0T})$ and using (35) ,(31) and hypothesis (iii) we get

$\alpha_i(A^{0T} + B^{0T}) = 0 \Rightarrow (A^{0T} + B^{0T}) = 0$

$\Rightarrow (A^0) = 0$,and = 0 , because > 0.

...(37)

(31) and (37) yield ...(38)

$\Rightarrow$

$$\{(f_{ix} - l_{0i}g_{ix}) - D(f_{i\dot{x}} - l_{0i}g_{i\dot{x}})\} \geq 0 ,$$

because $\alpha_i > 0$ . ...(39)

$\Rightarrow$. But $\omega_i x^{0T} = 0$.

This implies that $\alpha_i x^{0T}\{(f_{ix} - l_{0i}g_{ix}) - D(f_{i\dot{x}} - l_{0i}g_{i\dot{x}})\} = 0$

...(40)

(39) , (40) and (38) show that $(x_0(t), y_0(t), A^0 = 0, B^0 = 0, \lambda_0)$ satisfies the constraints of (MFD*) . i.e. $(x_0(t), y_0(t), R^0 = 0, S^0 = 0, \lambda_0)$ is feasible for (MFD*) are equal there.

$\mu_i y^{0T}\{(f_{iyy}A^{0T} - l_{0i}g_{iyy}B^{0T}) - D(f_{i\dot{y}}A^{0T} - l_{0i}g_{i\dot{y}}B^{0T})\} = 0$

$(x_0(t), y_0(t), A^0 = 0, B^0 = 0, \lambda_0)$ is efficient for (MFD*) .

If $(x_0(t), y_0(t), A^0 = 0, B^0 = 0, \lambda_0)$ is not properly efficient , then for some feasible $(u_i(t), v_i(t), A^0 = 0, B^0 = 0, \lambda_0)$ with

$$L_{0i} = \frac{\int_a^b f_i(t, u_i(t), \dot{u}_i(t), v_i(t), \dot{v}_i(t))\,dt - \int_a^b \frac{1}{2}\left[\begin{array}{l} R^T\{f_{ixx}(t, u_i(t), \dot{u}_i(t), v_i(t), \dot{v}_i(t)) \\ -D^2 f_{i\ddot{x}}(t, u_i(t), \dot{u}_i(t), v_i(t), \dot{v}_i(t))\}R \end{array}\right]dt}{\int_a^b g_i(t, u_i(t), \dot{u}_i(t), v_i(t), \dot{v}_i(t))\,dt + \int_a^b \frac{1}{2}\left[\begin{array}{l} S^T\{g_{ixx}(t, u_i(t), \dot{u}_i(t), v_i(t), \dot{v}_i(t)) \\ -D^2 g_{i\ddot{x}}(t, u_i(t), \dot{u}_i(t), v_i(t), \dot{v}_i(t))\}S \end{array}\right]dt}$$

,i =1,2,.,.,k,

and for some i , $L_{0i} - l_{0i} > M$ for any $M > 0$.

Since denominator at (t, x(t), $\dot{x}$(t), v(t), (t)) is bounded , it follows that

,

which contradicts weak duality .

Thus (x_0(t),y_0(t),A^0 =0,B^0 =0,λ_0) is properly efficient for (MFD*) .

A converse duality theorem is stated now. The proof is analogous to that of theorem 3.2.

Theorem 3.3

Let (x_0(t),y_0(t), l_0,λ_0) be a properly efficient solution for (MFD*). Fix for $\lambda=\lambda_0$ in (MFP*) and define

$$\frac{\int_a^b f_i(t,x_0,\dot{x}_0,y_0,\dot{y}_0)dt-\int_a^b \frac{1}{2}\left[A^T\{f_{iyy}(t,x_0,\dot{x}_0,y_0,\dot{y}_0)-D^2 f_{i\dot{y}\dot{y}}(t,x_0,\dot{x}_0,y_0,\dot{y}_0)\}A\right]dt}{\int_a^b g_i(t,x_0,\dot{x}_0,y_0,\dot{y}_0)dt+\int_a^b \frac{1}{2}\left[B^T\{g_{iyy}(t,x_0,\dot{x}_0,y_0,\dot{y}_0)-D^2 g_{i\dot{y}\dot{y}}(t,x_0,\dot{x}_0,y_0,\dot{y}_0)\}B\right]dt}=\frac{F_{i0}(x,y)}{G_{i0}(x,y)}=l_{i0}$$

i = 1,2,.,.,k.

Assume that

(I)

$$\sum_{i=1}^{k}\lambda_{0i}\int_a^b \varphi(t)^T([\tfrac{1}{2}\{(f_{ixxx}-D^2 f_{i\dot{x}\dot{x}\dot{x}})A^0-l_{0i}(g_{ixxx}-D^2 g_{i\dot{y}\dot{y}\dot{y}})-Df_{i\dot{x}}+D^2 f_{i\dot{x}\dot{x}}\}]$$
$$-D\tfrac{1}{2}[\{(f_{ixx\dot{x}}-D^2 f_{i\dot{x}\dot{x}\dot{x}}-Df_{i\dot{x}\dot{x}})A^0+(f_{ix\dot{x}}-Df_{i\dot{x}\dot{x}}-f_{i\dot{x}})\}-$$
$$l_{0i}\{(g_{ixx\dot{x}}-D^2 g_{i\dot{x}\dot{x}\dot{x}}-Dg_{i\dot{x}\dot{x}})B^0+(g_{ix\dot{x}}-Dg_{i\dot{x}\dot{x}}-g_{i\dot{x}})\}]$$
$$+D^2[\{-\tfrac{1}{2}(f_{i\dot{x}\dot{x}}+Df_{i\dot{x}\dot{x}\dot{x}})A^0+f_{i\dot{x}}\}-l_{0i}\{-\tfrac{1}{2}(g_{i\dot{x}\dot{x}}+Dg_{i\dot{x}\dot{x}\dot{x}})B^0+g_{i\dot{x}}\}]-D^3[-\{f_{i\dot{x}\dot{x}\dot{x}}A^0\}-\{-l_{0i}\{g_{i\dot{x}\dot{x}\dot{x}}B^0\}\}]\varphi(t))=0$$

implies $\phi(t)=0$, for all $t\in I$, and

(II) $\{\int_a^b(f_{1x}-l_{01}g_{1x})dt$, $\int_a^b(f_{2x}-l_{02}g_{2x})dt$,.,.,

$\int_a^b(f_{kx}-l_{0k}g_{kx})dt\}$ is linearly independent. If The invexity conditions of Theorem 3.1 are satisfied , then (x_0(t),y_0(t),l_0 ,λ_0) is properly efficient for (MFP*)

In order to present a better view of the concept of second order self-duality , we will consider here problems (MFP) and (MFD) instead of their equivalents (MFP*) and (MFD*) . Assume that x(t) and y(t) have the same dimensions , i.e. m = n the functions will said to be skew symmetric if f(t,x, $\dot{x}$,y,) = – f(t, y, ,x,) for all x and y with(t) and (t) are piece wise smooth , in the domain of f and the function will be called symmetric if g(t,x, ,y,) = g(t, y, ,x,) in the domain of g .

Theorem 3.4 (Self-Duality)

If f(t,x, ,y,) is skew symmetric and g(t,x, , y,) is symmetric , then (MFP) and (MFD) are self-dual . If (MFP) and (MFD) are dual problems , then with (x_0(t),y_0(t),λ_0 ,λ_0) also (y_0(t), x_0(t),λ_0 ,λ_0) is a joint optimal solution and the common optimal value is 0 .

Proof

With f skew symmetric

f_x(t,x, ,y,) = – f_y(t, y, ,x,)

f_y(t, y, ,x,) = – f_x(t,x, ,y,)

f (t,x, ,y,) = – f (t, y, ,x,)

f (t, y, ,x,) = – f (t,x, , y,)

and g is symmetric , we have

g_x(t,x, , y,) = g_y(t, y, , x,)

g_y(t, y, , x,) = g_x(t, x, ,y,)

g (t,x, ,y,) = g (t, y, , x,)

g (t, y, ,x,)= g (t,x, ,y,)

Expressing the dual problem (MFD) as a minimization problem and making use of the above relations , we have

Min. = =

$$\frac{F_i(u,v)}{G_i(u,v)}$$

$$u(a)=0=u(b),\ v(a)=0=v(b)$$

$$\int_a^b f_i(t,u(t),\dot{u}(t),v(t),\dot{v}(t))dt-\int_a^b\frac{1}{2}\left[R^T\{f_{iuu}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))-D^2 f_{i\dot{u}\dot{u}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))\}R\right]dt$$

$$\int_a^b g_i(t,u(t),\dot{u}(t),v(t),\dot{v}(t))dt+\int_a^b\frac{1}{2}\left[S^T\{g_{iuu}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))-D^2 g_{i\dot{u}\dot{u}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))\}S\right]dt$$

$$-Dg_{i\dot{u}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))\}F_i(u,v)]$$
$$+[\{f_{iuu}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))-D^2 f_{i\dot{u}\dot{u}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))\}RG_i(u,v)-\{g_{iuu}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))$$
$$-D^2 g_{i\dot{u}\dot{u}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))\}SF_i(u,v)]$$

≥0,

subject to

u(a) = 0 = u(b) , v(a) = 0 = v(b).

$$\dot{u}(a)=0=\dot{u}(b),\ \dot{v}(a)=0=\dot{v}(b)$$

≤0 ,

$$\int_a^b u(t)^T\sum_{i=1}^{k}\lambda_i[\{f_{iu}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))-Df_{i\dot{u}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))\}G_i(u,v)-\{g_{iu}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))$$
$$-Dg_{i\dot{u}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))\}F_i(u,v)]$$
$$+[\{f_{iuu}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))-D^2 f_{i\dot{u}\dot{u}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))\}RG_i(u,v)-\{g_{iuu}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))\ \geq 0$$
$$-D^2 g_{i\dot{u}\dot{u}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))\}SF_i(u,v)]$$

$$\lambda > 0,\ \lambda^T e = 1, t \in I .$$

Which is just the primal problem (MFP). Thus, if ($x_0(t), y_0(t), l_0, \lambda_0$) is an optimal solution of (MFD), then ($y_0(t), x_0(t), \lambda_0, \lambda_0$) is an optimal solution of (MFD) .

Since f is skew-symmetric and g is symmetric , respectively , we have

$l_{0i}(t, y_0, y_0, x_0, \quad) = -\, l_{0i}(t, x_0, \quad, y_0, \quad)$

$l_{0i}(t, x_0, \quad, y_0, \quad) = l_{0i}(t, y_0, \quad, x_0, \quad) = -\, l_{0i}(t, x_0, \quad, y_0, \quad)$

and so $l_{0i}(t, y_0, \quad, x_0, \quad) = l_{0i}(t, x_0, \quad, y_0, \quad) = 0$.

DISCUSSION

Particulars cases and remarks in applications

(i) If we take the static case then this is the recent work done in[28].

(ii) If we take the static case and the function f to be real and differentiable with F-Convexity and $F_0=F_1$, then this is an earlier work[3].

(iii) If we first order case then this is work done in[31].

(iv) If we take the static case and the function f to be real and convex / concave then this is an earlier work by Gulati and Ahmad[34].

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