A note on common fixed point theorem in Hilbert Space

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ABSTRACT

In this paper the authors studied the problem of Sayyed and Badshah[8] and prove common fixed point theorem in Hilbert Space. In recent years Rashwan and Sadik[5], Malnge[3], Berinde[1], Rashwan[4], Song and Chen[11], Cric, Ume and Khan[2] have studied the convergence of iterations to common fixed point for a pair of mappings.

Rhoades[6-7], proved the mappings T satisfying certain contractive condition, if the sequences of Mann iterates converged it converges to a fixed point of T. Sayyed and Badshah[9-10] proved generalized contractive type mapping in Hilbert Space. AMS (2000) Subject Classifications: Primary 47H10, Secondary 54H25

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INTRODUCTION

Let X be a Banach space and C be a non-empty subset of X. Let T_1, T_2 : C $\rightarrow$ C be two mappings. The iteration scheme called I-Scheme is defined as follows:

$$x0 \in C \qquad ...(1)$$

...(2)

...(3)

In the Ishikawa scheme $\{\alpha_{2n}\}$, $\{\beta_{2n}\}$ satisfy $0 \le \alpha_{2n} \le \beta_{2n} \le 1$, for all n. and

. In this paper we shall make the assumption that

i) $0 \le \alpha_{2n} \le \beta_{2n} \le 1$, for all n,

[illegible]

ii) $x_{2n+2} = (1-\alpha_{2n})x_{2n+1} + \alpha_{2n} T_1 y_{2n+1}, n \ge 0$ and

iii)

We know that Banach space is Hilbert if and only if its norm satisfies the parallelogram law i.e. every x, y $\in$ x (Hilbert space).

...(4)

which implies

$$\|x+y\|^2 \le 2\|x\|^2 + 2\|y\|^2 \qquad ...(5)$$

We often use this inequality throughout the result.

Below we prove the result concerning the existence of common fixed point of pairs of mappings satisfying the concentration of the type:

$$\|Tx - Ty\|^2 \le K \max\left\{\frac{\|x-Tx\|^2[1+\|y-Ty\|]}{1+\|x-y\|^2}\right\}, \left\{\frac{1+\|y-Ty\|^2[1+\|y-Tx\|^2]}{1+\|x-y\|^2}\right\}$$

$$\frac{1}{2}\left[\|x-Tx\|^2+\|y-Tx\|^2\right]$$

Theorem

Let X be a Hilbert space and C to be a closed convex, subset of X. Let T_1 and T_2 be two sets of mapping satisfying

$$\frac{1}{2}\left[\|x-T_1x\|^2+\|y-T_2y\|^2\right]\frac{1}{2}\left[\|x-T_2x\|^2+\|y-T_1x\|^2\right]$$

...(8)

Where $0 \le K < \frac{1}{4}$. If there exists a point x0 such that the I scheme for T_1 and T_2 defined by (2) and (3), converges to a point p, then p is a common fixed point of T_1 and T_2.

Proof:

It follows from (2) that $x_{2n+1} - x_{2n} = \alpha_{2n} \{T_2y2n - x_{2n})$. Since $x_{2n} \to p$, $\|x_{2n+1} - x_{2n}\| \to 0$. Since $\{\alpha_{2n}\}$ is bounded away from zero, $\|T_2y_{2n}-x_{2n}\| \to 0$.

It also follow that $\| p - T_2y_{2n}\| \to 0$. Since T_1 and T_2 satisfies (7), we have

$$\|Tx-Ty\|^2 \le \mathrm{K}\max\left\{\frac{\|x-T_1x_{2n}\|^2\left[1+\|y_{2n}-T_2y_{2n}\|\right]}{1+\|x_{2n}-T_2y_{2n}\|^2}\right\},\left\{\frac{1+\|y_{2n}-T_2y_{2n}\|^2\left[1+\|x_{2n}-T_1x_{2n}\|^2\right]}{1+\|x_{2n}-y_{2n}\|^2}\right\}$$

$$\frac{1}{2}\left[\|x_{2n}-T_1x_{2n}\|^2+\|y_{2n}-T_2y_{2n}\|^2\right]\frac{1}{2}\left[\|x_{2n}-T_2y_{2n}\|^2+\|y-T_1x\|^2\right]$$

Now

$$\begin{aligned}\|y2n-x2n\|^2 &= \|\beta_{2n}T_1x_{2n}+(1-\beta_{2n})x_{2n}-x_{2n}\|^2\\ &= \|\beta_{2n}T_1x_{2n}+x_{2n}-\beta_{2n}x_{2n}-x_{2n}\|^2\\ &= \|\beta_{2n}(T_1x_{2n}-x_{2n})\|^2\\ &\le 2\beta_{2n}^2\|T_1x_{2n}+T_2y_{2n}\|^2+2\beta_{2n}{}^2x(T_2y_{2n}-x_{2n})\|^2\\ &\le 2\|T_1x_{2n}+T_2y_{2n}11^2+2(T_2y_{2n}-x_{2n})\|^2\end{aligned}$$

...(9)

$$\begin{aligned}\|y_{2n}-T_1x_{2n}\|^2 &= \|\beta_{2n}T_1x_{2n}+(1-\beta_{2n})x_{2n}-T1x_{2n}\|^2\\ &= \|(1-\beta_{2n})(x_{2n}-T_1x_{2n})\|^2\\ &= (1-\beta_{2n})^2\|(x_{2n}-T_1x_{2n})\|^2\\ &\le (1-\beta_{2n})^2\left\{\|(x_{2n}-T_2y_{2n})+(T_2y_{2n}-T_1x_{2n})\|^2\right\}\\ &\le 2(1-\beta_{2n})^2\|x_{2n}-T_2y_{2n}\|^2+2\|(1-\beta_{2n})^2\|T_2y_{2n}-T_1y_{2n})\|^2\\ &\le 2^2\|x_{2n}-T_2y_{2n}\|^2+2\|T_2y_{2n}-T_1x_{2n})\|^2\end{aligned}$$

...(10)

From (9), (10), (11) and (8) can be written as

$$\|T_1x_{2n}-T_2Y_{2n}\|^2 \le K\max\left\{\frac{\|x_{2n}-T_1x_{2n}\|^2\left[1+2\|T_1x_{2n}-T_2Y_{2n}\|^2+2\|x_{2n}-T_2Y_{2n}\|^2\right]}{\left[1+2\|T_1x_{2n}-T_2Y_{2n}\|^2+2\|T_2y_{2n}-x_{2n}\|^2\right]}\right\}$$

$$\frac{\left[1+2\|T_1x_{2n}-T_2Y_{2n}\|^2+2\|x_{2n}-T_2Y_{2n}\|^2\right]\left[1+\|x_{2n}-T_1Y_{2n}\|^2\right]}{\left[1+2\|T_1x_{2n}-T_2Y_{2n}\|^2+2\|T_2y_{2n}-x_{2n}\|^2\right]}$$

$$\left(\|x_{2n}-T_1Y_{2n}\|\right)\left[1+2\|T_1x_{2n}-T_2Y_{2n}\|^2+2\|T_2y_{2n}-x_{2n}\|^2\right]$$

[illegible]

$$\frac{2\left[1+2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|T_2y_{2n}-x_{2n}\|^2\right]\left[1+\|x_{2n}-T_1y_{2n}\|^2\right]}{\left[1+2\|T_1x_{2n}-T_2Y_{2n}\|^2+2\|x_{2n}-T_2y_{2n}\|^2\right]}$$

$$\frac{1}{2}\left(\|\mathrm{x}_{2n}-T_1x_{2n}\|^2+2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|\mathrm{x}_{2n}-T_2y_{2n}\|^2\right),$$

$$\frac{1}{2}\left(\|\mathrm{x}_{2n}-T_2y_{2n}\|^2+2\|x_{2n}-T_2y_{2n}\|^2+2\|\mathrm{T}_2\mathrm{y}_{2n}-T_1x_{2n}\|^2\right),$$

$$2\|\mathrm{T}_1\mathrm{x}_{2n}-T_2y_{2n}\|^2+2\|\mathrm{T}_2y_{2n}-x_{2n}\|^2\Big)\Big\}$$

$$\le K\left\{2\|\mathrm{T}_1\mathrm{x}_{2n}-T_2y_{2n}\|^2+2\|\mathrm{T}_2y_{2n}-x_{2n}\|^2\Big)\right\}$$

Thus, $\|\mathrm{T}_1\mathrm{x}_{2n}-T_2y_{2n}\|^2 \le \frac{2K}{1-2K}\|\mathrm{T}_2y_{2n}-x_{2n}\|^2$

Taking the Lim as $n\to\infty$, we get $\|\mathrm{T}_1\mathrm{x}_{2n}-T_2y_{2n}\|^2\to 0$. It folows that

and

$$\|P-T_1x_{2n}\|^2 \le \|\mathrm{P}\text{-}\mathrm{x}_{2n}\|^2+2\|x_{2n}-T_1y_{2n}\|^2 \to 0 \text{ as } n\to\infty$$

If x_{2n}, P satisfies (7), we have

$$\frac{\left(1+\|P-T_2P\|^2\right)\left(1+\|x_{2n}-T_1x_{2n}\|^2\right)}{1+\|x_{2n}-p\|}, \frac{\|x_{2n}-T_1x_{2n}\|^2\left(1+\|x_{2n}-P\|^2\right)}{1+\|P-T_2p\|}$$

$$\frac{\|P-T_2P\|\left(1+\|x_{2n}-T_1x_{2n}\|^2\right)}{1+\|P-T_2P\|}, \frac{1}{2}\left[\|x_{2n}-T_1x_{2n}\|^2+\|x_{2n}-P\|^2\right]$$

$$\left.\frac{1}{2}\left[\|x_{2n}-T_2P\|^2+\|P-T_1x_{2n}\|^2\right], \|x_n-P\|^2\right\}$$

$$\frac{\|x_{2n}-T_1x_{2n}\|^2\left(1+\|x_{2n}-P\|^2\right)}{1+2\|p-x_{2n}\|^2+2\|x_{2n}-T_2P\|^2}, \frac{\|x_{2n}-P\|^2\left(1+\|x_{2n}-T_1x_{2n}\|^2\right)}{1+2\|p-x_{2n}\|^2+2\|x_{2n}-T_2P\|^2}$$

$$\frac{1}{2}\left(\|x_{2n}-T_1x_{2n}\|^2+2\|P-x_{2n}\|^2+2\|x_{2n}-T_2P\|^2\right),$$

$$\left.\frac{1}{2}\left[\|x_{2n}-T_2P\|^2+2\|P-T_1x_{2n}\|^2\right), \|x_{2n}-P\|^2\right\}$$

Taking the Lim as n n $\to \infty$, we get

$$\Rightarrow$$

$$\|T_1x_{2n}-T_2p\|^2 \to 0.$$

Fianally

$$\|P-T_2P\|^2 \le 2\|P-T_1x_{2n}\|^2+2\|T_1x_{2n}-T_2P\|^2 \to 0\, n \to \infty$$

Showing that p = T_2p.

Similarly, we can prove that p = T_1 p.

Thus p is the common fixed point of T_1 and T_2. This completes the proof.

Letting $T_1 = T_2 = T$ in above theorem, we obtain the following corollary.

Corollary

Lex X be a Hilbert Space and C be a closed conved subset of X. Let T be a self-mapping

satisfying (6), where if there exists a

point x_0 such that the I - scheme for T defined by

[illegible]

$$y_n = \beta_n T x_n + (1-\beta_n) x_n,\ n \ge 0$$

converges to a point p, then p is the fixed point of T.

In the I - Scheme $\{\alpha_n\}$, $\{\beta_n\}$, satisfy $0 \le \alpha_{2n} \le \beta_{2n} \le 1$, for all n. . Assuming that

i) $0 \le \alpha_n \le \beta_n \le 1$, for all n,

ii) and

iii)

The proof is similar to above theorem, hence we omit the details

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