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Fixed point theorems for the pair of coincidentally commuting mappings in d-metric space

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ABSTRACT

In this paper common fixed point of pair of coincidentally commuting mappings in D-metric spaces have been proved.

Key words: Common fixed point, D-metric spaces, coincidentally commuting mappings.

INTRODUCTION

Dhage[1,2,3] introduced the concept of D-metric space and proved several results. Rhoades[4] also established interesting results on D-metric spaces. Jungck[5,6] introduced a more general concept known as compatible mapping in metric spaces. Ume[7] proved non convex minimization theorem in D-metric spaces.

Definition 1

If $\rho(X)$ is a collection of all non-empty bounded subsets of a D-metric space (X,D) and for $A,B,C \in \rho(X)$, let $H(A,B,C) = \sup\{D(a,b,c) : a \in A, b \in B, c \in C\}$,then (1) $H(A, B, C) \geq 0$ and H(A B,C) = 0 implies A=B=C, with a singleton, further if A=B=C, then H(A,B,C) = perimeter of the largest triangle contained in the set A > 0, otherwise A is singleton,

(1) $H(A,B,C) = H(B,C,A) = H(C,A,B)$,
(2) $H(A,B,C) \leq H(A,B,E) + H(A,E,C) + H(E,B,C)$

Definition 2

A point $x_o \in X$ is said to be fixed point if $Tx_o = x_o$ i.e. a point which remain in variant under a transformation T is called a fixed point.

Coincidentally Commuting Mappings

The commutativity of pairs of maps is vital for proving the common fixed point theorems and Jungck[5] first used it in the ordinary metric space.

Definition 3

Two maps $f,g : X \to X$ are said to be commutative or commuting if $fg(x) = gf(x)$ for all $x \in X$.

In an ordinary metric space (X,d), Sessa[9] first introduce a weaker version of the commutativity for a pair of self maps of X as follows :

Definition 4

Two maps $f,g:(X,d) \to (X,d)$ are called weakly commutative or weakly commuting if $d(fg(x), gf(x)) \leq d(fx,gx)$ for all $x \in X$.

It is shown in research paper of Sessa[8] that a weakly commuting pair of maps in metric space is commuting, but the converse may not be true. In the following we list a few weaker versions of the commutativity for pairs of maps in metric spaces appeared in the earlier literatures.

Definition 5

Jungck[6], Two maps $f,g:(X,d) \to (X,d)$ are said to be compatible if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X satisfying $\lim(fx_n, gx_n) = 0$. It has been shown in Jungck[9] that every weakly commuting pair of maps is compatible, but the reverse implication may not hold.

Definition 6

Two maps $f,g : X\to X$ are said to be coincidentally commuting or coincidence preserving if they commute at coincidence points.

Thus we have a one-way implication, namely, commuting maps $\Rightarrow$ weakly commuting maps $\Rightarrow$ compatible maps $\Rightarrow$ coincidentally commuting maps.

Example 1

Let X = R and define $f,g : R\to R$ by $f(x)= x/2$ and $g(x) = x^2$ for $x \in R$. Clearly there are two coincidence points for the maps f and g in R namely 0 and ½. Note that f and g commute at 0, i.e. $fg(0) = gf(0)$, but $fg(½)=1/8\neq gf(½)$ and so f and g are not coincidentally commuting on R.

Definition 7

Let $S,T : X \to X$, then the orbit of S and T at a point $x \in X$ is a set (3) $O(S,T;x) = \{x, Sx, TSx, STSx, \ldots\}$

Then the D-metric space X is said to be (S,T)-orbitally bounded if the orbit O(S,T;x) is bounded for each $x \in X$. The orbit O(S,T;x) is called complete if every D-Cauchy sequence in O(S,T,x) converges to a point in X. A (S,T)-orbitally complete D-metric space X is one in which every orbit O(S,T;x), $x \in X$, is complete.

Useful lemma in the sequel

Lemma 1. (D-Cauchy Principle)

Let $\{x_n\} \subseteq X$ be bounded with D-bound k satisfying (4) $D(x_n,x_{n+1},x_m) \leq a^n k$ for all $m > n \in N$ and $0 \leq \alpha \leq 1$, then $\{x_n\}$ is D-Cauchy.

Lemma 2 (D-Cauchy Principle)

Let $\{x_n\} \subseteq X$ be bounded with D-bound k satisfying

(5) $D(x_n, x_{n+1}, x_m) \leq \phi^n k$

for all $m > n \in N$, where $\phi:R^+ \to R^+$ satisfies

$$\sum_{n=1}^{\infty} \phi^n(t) < \infty$$

for each $t \in R^+$. Then $\{x_n\}$ is D-Cauchy.

Lemma 3

If x is (X, T)-orbitally bounded D-metric space and $\{x_n\} \subseteq 0(S,T;x)$, $x \in X$ satisfying

(6) $D(x_n, x_{n+1}, x_m) \leq \phi^n(t)$, for all $m > n \in N$,

where $\phi:R^+ \to R^+$ satisfies $\sum_{n=1}^{\infty} \phi^n(t) < \infty$

for each $t \in R^+$. Then $\{x_n\}$ is D-Cauchy.

Let ϕ denotes the class of all functions $\phi:R^+ \to R^+$ satisfying

(7) ϕ is continuous,

(8) ϕ is nondecreasing,

(9) $\phi(t) < t$ for $t > 0$,

(10) $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t \in R^+$

A member ϕ of class Φ is called a control or contraction function and commonly used control function is $\phi(t) = \alpha t$, $0 \leq \alpha < 1$. We need the following lemma in the sequel.

Lemma 4

If $\phi \in \Phi$, then $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$ and $\phi^n(0) = 0$ for each $n \in N$.

Below we prove the main result of this paper.

Theorem 1. Let $S, T : X \to X$ and let X be (S,T)-orbitally complete and (S,T)-orbitally bounded D-metric space and suppose that

$$D(Sx,Ty,z) \leq \phi\left(\alpha \frac{D(x,Sx,z)^2 + D(y,Ty,z)^2}{D(x,Sx,z) + D(y,Ty,z)} + \beta D(x,y,z)\right)$$

for all $x,y \in X$ and $z \in O(S, T;x) \cup O(T, S;y)$, where $0 \leq 2\alpha+\beta < 1$ and $\phi \in \Phi$. Then S and T have a unique common fixed point.

Proof

Let $x \in X$ be arbitrary and define a sequence $\{x_n\} \subset X$ by

(12) $x_0=x,\ x_{2n+1}= Sx_{2n},\ x_{2n+2} = Tx_{2n+1},\ n \geq 0.$

We show that $\{x_n\}$ is D-Cauchy. Now for any $m \geq 2$, by (11) we have

$$D(x_1, x_2, x_m) = D(Sx_0, Tx_1, x_m)$$

$$\leq \phi\ \alpha\ \frac{D(x_0,x_1,x_m)^2 + D(x_1,x_2,x_m)^2}{D(x_0,x_1,x_m) + D(x_1,x_2,x_m)} + \beta D(x_0,x_1,x_m)$$

$$\leq \phi(\alpha(D(x_0, x_1, x_m)+D(x_1,x_2, x_m)+\beta D(x_0, x_1, x_m))$$
$$\leq \alpha(D(x_0, x_1, x_m)+ \alpha D(x_1, x_2, x_m) + \beta D(x_0, x_1, x_m))$$

(13) $D(x_1, x_2, x_m) \leq \dfrac{\alpha+\beta}{1-\alpha}\ D(x_0, x_1, x_m)$

where k is a D-bound of O(S,T;x).

Similarly for $m \geq 3$, we get

$$D(x_2, x_3, x_m) = D(Sx_2, Tx_1, x_m)$$

$$\leq \phi\ \alpha + \frac{(D(x_2, x_3, x_m)^2 + D(x_1, x_2, x_m)^2}{D(x_2, x_3, x_m) + D(x_1, x_2, x_m)}\ \beta D(x_1, x_2, x_m)]$$

$$\leq \phi[\alpha(D(x_2, x_3, x_m)+ D(x_1, x_2, x_m) + \beta D(x_1, x_2, x_m)]$$
$$\leq [\alpha D(x_2, x_3, x_m)+ \alpha D(x_1, x_2, x_m) + \beta D(x_1, x_2, x_m)]$$
$$D(x_2, x_3, x_m) \leq (\alpha + \beta/1\text{-}a)D(x_1, x_2, x_m) \leq (\alpha + \beta/1\text{-}\alpha)^2\ D(x_0, x_1, x_m).$$

In general for $m \geq n + 1$, one has

(14) $D(x_n, x_{n+1}, x_m) \leq (\alpha + \beta/1\text{-}a)^n\ D(x_0, x_1, x_m)$

which implies that $\{x_n\}$ is D-Cauchy, Since X is (S, T) orbitally complete,

$\lim_{n\to\infty} x_n = u$ exists. We show that u is a common fixed point of S and T.

Now $D(u, Tu, u) = \lim_{n\to\infty} D(x_{2n},Tu,u)$

$$= \lim_{n\to\infty} D(Sx_{2n+1},Tu,u)$$

$$\leq \lim_{n\to\infty} \phi\ \alpha\ \frac{D(x_{2n}, Sx_{2n}, u)^2 + D(u, Tu, u)^2}{D(x_{2n}, Sx_{2n}, u) + D(u, Tu, u)} + \beta(x_{2n}, u, u)$$

$$\leq \phi(0 + \alpha D(u, Tu, u) + 0) \leq \phi(\alpha D(u, Tu, u)) < \alpha D(u, Tu, u)$$

(15) $(1\text{-}\alpha)\ D(u, Tu, u) < 0,$

which is possible only when u = Tu. Again we get,

$$D(u, Su, u) = D(Su, u, u) = D(Su, Tu, u)$$

$$\leq \phi\ \alpha\ \frac{D(u, Su, u)^2 + D(u,u,u)^2}{D(u, Su, u) + D(u,u,u)} + \beta(u,u,u)$$

$$\leq \phi[\alpha D(u, Su, u)] \leq \alpha D(u, Su, u)$$

(16) and so u = Su since $\phi \in \Phi$.

Thus u is a common fixed point of S and T. To prove uniqueness, let v(¹u) be another common fixed point of S and T. Then $D(u, u, v) \neq 0$ and we get $D(u, v, v) = D(Su, Tv, v)$

$$\leq \phi\ \alpha\ \frac{D(u,Su,v)^2 + D(v,Tv,v)^2}{D(u,Su,v) + D(v,Tv,v)^2} + \beta D(u,v,v)$$

$$\leq \alpha D(u, u, v)$$

Again interchanging the role of u and v we obtain $D(v, u, u) \leq \phi(D(v, v, u))$.

It follows that $D(u, v, v) \leq \phi^2(D(u, v, v))$.

Which is a contradiction and hence u = v. This completes the proof.

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